UNIFORM DECAY RATE ESTIMATES FOR SCHRÖDINGER AND PLATE EQUATIONS WITH NONLINEAR LOCALLY DISTRIBUTED DAMPING

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ABSTRACT. On a compact \( n \)-dimensional Riemannian manifold \((M, g)\), we establish uniform decay rate estimates for the linear Schrödinger and plate equations subject to an internal nonlinear damping locally distributed on the manifold. Our approach can be also employed for other equations provided that inverse inequality for the linear model occurs. In the particular case of the wave equation, where the well known geometric control condition (GCC) is equivalent to the observability inequality, our method generalizes the results due to Cavalcanti et. al. [9] [10] regarding the optimal choice of dissipative regions.

1. INTRODUCTION

This paper is addressed to the stabilization of Schrödinger and plate equations with nonlinear damping locally distributed:

\[
\begin{aligned}
\begin{cases}
iv_t + \Delta v + i a(x) g(v) &= 0 \quad \text{in } M \times (0, \infty) \\
y &= 0 \quad \text{on } \partial M \times (0, \infty) \\
y(0) &= y_0 \quad \text{in } M,
\end{cases}
\end{aligned}
\]

(1.1)

\[
\begin{aligned}
\begin{cases}
v_{tt} + \Delta^2 v + a(x) g(v_t) &= 0 \quad \text{in } M \times (0, \infty) \\
v &= \Delta v = 0 \quad \text{on } \partial M \times (0, \infty) \\
v(0) &= y_0, \quad v_t(0) = y_1 \quad \text{in } M,
\end{cases}
\end{aligned}
\]

(1.2)

where \((M, g)\) is \( n \)-dimensional compact Riemannian manifold with smooth boundary.

The non-negative function \( a(\cdot) \), responsible by the localized dissipative effect, satisfy the following condition:

\[
a \in L^\infty(M); a(x) \geq a_0 > 0 \text{ in } \omega \subset M,
\]

(1.3)

where \( \omega \) is an open subset of \( M \) properly contained in \( M \).

Considering \( g \equiv 0 \), that is, when (1.1) and (1.2) are linear, we shall assume the main assumption:

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Assumption 1.1. There exist \((ω,T_0)\), \(T_0 > 0, ω ⊂⊂ M\), such that the following inverse inequalities hold:

(1.4) \[ \|y_0\|_{L^2(M)}^2 \leq C \int_0^T \int_ω |y(x,t)|^2 \, dx \, dt, \]

regarding problem (1.1),

(1.5) \[ \|y_1\|_{L^2(M)}^2 + \|\Delta y_0\|_{L^2(M)}^2 \leq C \int_0^T \int_ω |∂_t y(x,t)|^2 \, dx \, dt, \]

concerning problem (1.2),

for some \(C = C(ω,T_0)\) and for all \(T > T_0\).

We shall discuss in what follows for some examples where the conditions (1.4) and (1.5) occur. A sufficient condition for (1.4) or (1.5) to hold was found by Lebeau [28] (see also [15]). It is the well known Geometric Control Condition (GCC) due to Bardos, Lebeau and Rauch [2], namely:

(1.6) Assume that the geodesics of \(\overline{M}\) have no contact of infinite order with \(∂M\).

Let \(ω\) be an open subset of \(M\) and suppose that there exists \(T_0 > 0\) such that every geodesic traveling at speed 1 and issued at \(t = 0\) meets \(\overline{ω}\) in a time \(t < T_0\).

However, this condition is not necessary in general, as follows from the works of Jaffard [17] or Burq and Zworski [7]. For instance, when \(M = \mathbb{T}^2\) the two-dimensional torus flat equipped with the flat metric, Jaffard [17] proved the following result.

Theorem 1.1. Let \((M,g) = (\mathbb{T}^2,\text{flat})\). Given \(T > 0\) and any open set \(ω \subset \mathbb{T}^2\) there exists a constant \(C > 0\) such that the observability inequality estimate (1.4) (or (1.5)) holds.

The proof of the above theorem contained in Jaffard [17] relies on results on pseudo-periodic functions due to Kahane. Subsequently a new proof to theorem (1.1) has been given by Macià [31] making use of microlocal arguments and his proof is based on the structure result for semiclassical measures for the Schrödinger flow on the torus.

Figure 1. Exemple where the GCC does not hold. The geodesic \(γ\) does not meet \(ω\)

Theorem 1.1 states that for Schrödinger or plate equations there exist some manifolds \((M,g)\) where the Geometric Control Condition (GCC) is not a necessary and sufficient condition to establish the observability inequality estimate (see figure 1). Nevertheless, it has been proved by Macià [31] that (1.6) is equivalent to (1.4) when \((M,g)\) has periodic flow (also known as Zoll manifolds).
Assumption 1.2. The following assumptions are made on the function $g$, which has borrowed from Lasiecka and Triggiani [25]:

\((H_1)\)

(i) $g : \mathbb{C} \to \mathbb{C}$ is continuous, $g(0) = 0$.

(ii) $g$ is the sub-differential of $J_1$, that is, $g(z) = \partial J_1(z)$, where $J_1 : \mathbb{C} \to \mathbb{R} = (-\infty, +\infty)$ is a lower semicontinuous, convex proper function.

(iii) $\Re \{(g(z) - g(v))(z - v)\} \geq 0$, $\forall z, v \in \mathbb{C}$.

(iv) $\Im \{g(z)\overline{z}\} \equiv 0$. Note that for $v = 0$, we have from (iii) that

$$\Re \{g(z)\overline{z}\} \geq 0,$$

which implies in view of (iv) that

$$\Re \{g(z)\overline{z}\} = g(z)\overline{z} \geq 0,$$

and, consequently,

$$g(z)\overline{z} = |g(z)|.$$

\((H_2)\) There exist $m, c > 0$ such that

(a) $m |z|^2 \leq g(z)\overline{z}$, if $|z| \leq 1$.

(b) $|g(z)| \leq c |z|$, if $|z| \geq 1$.

There is a reasonable literature with regard to the issue of control and stabilization of Schrödinger and plate equations. Among the numerous work we would like to mention the following: [3], [5], [6], [11], [12], [15], [18], [19], [17], [21], [22], [23], [24], [25], [26], [27], [28], [33], [32], [35], [37], [38], [39], [40], [41] and a list a references therein. As far as we are concerned the majority of papers in the literature regarding the stabilization of Schrödinger (or plate) equation subject to locally distributed damping treat the linear or the nonlinear case (focusing or defocusing equation) but the local dissipation possesses a linear character, that is, $g(z) = z$. When $g$ is nonlinear we are aware the paper [14] where the authors prove the polynomial decay for the energy of solutions of a plate equation of Bernoulli-Euler type with a nonlinear localized damping term.

The main goal of the present paper is to present a method to treat the asymptotic stability for linear equations subject to non-linear dissipations $a(\cdot)g(z)$. Despite the proof to be simple it is very effective for treating linear equations subject to non-linear dissipations. In what follows we shall give the main idea for the specific case of the Schrödinger model. Note that the solution $y$ of problem (1.1) can be written as a sum $y = \varphi + z$ of the following linear problems:

\[
\begin{align*}
\begin{cases}
    i\partial_t \varphi + \Delta \varphi = 0 & \text{in } \mathcal{M} \times (0, \infty) \\
    \varphi = 0 & \text{on } \partial \mathcal{M} \times (0, \infty) \\
    \varphi(0) = y_0 & \text{in } \mathcal{M}
\end{cases}
\quad \text{and} \quad
\begin{cases}
    i\partial_t z + \Delta z = -ia(x)g(y) & \text{in } \mathcal{M} \times (0, \infty) \\
    z = 0 & \text{on } \partial \mathcal{M} \times (0, \infty) \\
    z(0) = 0 & \text{in } \mathcal{M}.
\end{cases}
\end{align*}
\]

Indeed, denoting $w = \varphi + z$, then $w$ satisfies

\[
\begin{cases}
    i\partial_t w + \Delta w = -ia(x)g(y) & \text{in } \mathcal{M} \times (0, \infty) \\
    w = 0 & \text{on } \partial \mathcal{M} \times (0, \infty) \\
    w(0) = y_0 & \text{in } \mathcal{M}.
\end{cases}
\]
Setting \( v = y - w \) then \( v \) is solution of
\[
\begin{aligned}
& i \partial_t v + \Delta v = 0 \quad \text{in } \mathcal{M} \times (0, \infty) \\
& v = 0 \quad \text{on } \partial \mathcal{M} \times (0, \infty) \\
& v(0) = 0 \quad \text{in } \mathcal{M},
\end{aligned}
\]
which implies that \( v \equiv 0 \), or, in other words, that \( y = w \), as we desired. Having the above in mind, let us assume that the observability inequality for the linear problem holds, that is,
\[
||y_0||_{L^2(\mathcal{M})}^2 \leq C \int_0^T \int_{\omega} |\varphi(x, t)|^2 \, dx \, dt,
\]
for some \( C > 0 \) and for all \( T > T_0 \). Defining \( E_y(t) := \frac{1}{2} ||y(t)||_{L^2(\mathcal{M})}^2 \) our main task is to prove that the following main inequality holds:
\[
E_y(T) \leq CT \int_0^T \int_{\mathcal{M}} a(x)(||y||^2 + |g(y)|^2) \, dx \, dt.
\]
Assuming that (1.8) takes place and proceeding verbatim as considered in Lasiecka and Triggiani [25] the solution of problem (1.1) satisfies the following decay rate
\[
E_y(t) \leq S \left( \frac{1}{T_0} \right) E_y(0) \searrow 0, \quad \text{for all } t \geq T_0, \ t \to \infty,
\]
where the scalar function \( S(t) \) (nonlinear contraction) is the solution of the following ODE:
\[
\frac{d}{dt} S(t) + q(S(t)) = 0, \quad S(0) = E_y(0),
\]
where the function \( q \) is defined in Lasiecka and Triggiani [25] (see (2.12) on page 492). Another important issue is the identity of the energy to problem (1.1):
\[
E_y(t) + 2 \int_s^t \int_{\mathcal{M}} g(y) \bar{y} \, dx \, dt = E_y(s), \quad \text{where then } E(t) \leq E(s), \ t \geq s \geq 0.
\]
The last important step in the proof is the well known result which states that the map \( \{z_0, f\} \mapsto z \) that associates the initial data \( \{z_0, f\} \in L^2(\mathcal{M}) \times L^1(0, T; L^2(\mathcal{M})) \) to the unique solution of the linear problem
\[
\begin{aligned}
& i \partial_t z + \Delta z = f \quad \text{in } \mathcal{M} \times (0, \infty) \\
& z = 0 \quad \text{on } \partial \mathcal{M} \times (0, \infty) \\
& z(0) = z_0 \quad \text{in } \mathcal{M},
\end{aligned}
\]
is linear and continuous, that is,
\[
||z||_{L^\infty(0,T; L^2(\mathcal{M}))} \leq ||z_0||_{L^2(\mathcal{M})} + ||f||_{L^1(0,T; L^2(\mathcal{M}))}.
\]
Thus, combining (1.3), (1.7), (1.11) and (1.12) and having in mind that \( z_0 = 0 \) and \( f = -ia(x)g(y) \) we deduce

\[
E_y(T) = \frac{1}{2} \| y(T) \|_{L^2(M)}^2 \leq \frac{1}{2} \| y_0 \|_{L^2(M)}^2 \leq C \int_0^T \int_\omega |\varphi(x,t)|^2 \, dx \, dt
\]

\[
\leq \tilde{C} \int_0^T \int_\omega a(x)|\varphi(x,t)|^2 \, dx \, dt
\]

\[
\leq \hat{C} \int_0^T \int_\omega a(x) [ |y(x,t)|^2 + |z(x,t)|^2 ] \, dx \, dt
\]

\[
\leq C' \int_0^T \int_\omega a(x) [ |y(x,t)|^2 + |g(y(x,t))|^2 ] \, dx \, dt,
\]

which establishes the desired in (1.8).

From the explanations above the main ingredients to establish uniform decay rate estimates to problem (1.1) (or (1.2)) are the observability inequality for the linear problem \( \varphi \) above mentioned and the wellposedness to problem (1.1) (or (1.2)). However, the observability inequality has been established in some important works (see [17], [28], [33], [31]). Consequently it remains to prove the wellposedness to problem (1.1) (or (1.2)), which is not an easy task. For this purpose we shall use nonlinear semigroup theory.

From the above we are in a position to outline our main result.

**Theorem 1.2.** Assume that (1.3) and Assumption 1.1 and Assumption 1.2 hold. Then, problem (1.1) (respec. (1.2)) possesses a unique generalized solution which satisfies the decay rate estimate given in (1.9).

Our paper is organized as follows. Sections 2 and 3 are, respectively, devoted to the proof of the wellposedness of equations (1.1) and (1.2) as well as the proof of the main inequality (1.8) which allows to conclude the proof of Theorem 1.2. In section 3 we give some examples of decay rate estimates.

## 2. Schrödinger Equation

In this section we shall study problem (1.1).

### 2.1. Notation.

We consider the space \( L^2(M) \) of complex valued functions on \( M \). It is a real Hilbert space when endowed with the inner product

\[
(y, z)_{L^2(M)} = \text{Re} \int_M y(x) \overline{z(x)} \, dx
\]

with the corresponding norm

\[
\| y \|_{L^2(M)}^2 = (y, y)_{L^2(M)}.
\]

We consider also the Sobolev space \( H^1_0(M) \) endowed with scalar product

\[
(y, z)_{H^1_0(M)} = (\nabla y, \nabla z)_{L^2(M)}.
\]
We shall write problem (1.1) as a Cauchy problem in $L^2(\mathcal{M})$ the form:

\[
\begin{aligned}
\partial_t y + Ay + By &= 0 \\
y(0) &= y_0,
\end{aligned}
\]

where $A : D(A) = H^1_0(\mathcal{M}) \cap H^2(\mathcal{M}) \subset L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is defined by $Ay = -i\Delta y$, for $y \in D(A)$ (with $D(A)$ dense in $L^2(\mathcal{M})$) and $B : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is defined by $By = a(\cdot)g(y)$, for $y \in L^2(\mathcal{M})$. We shall prove that $\mathcal{A} + B$ is self-adjoint, consequently $\mathcal{A}$ is skew-adjoint and, therefore, $\mathcal{A}$ is maximal monotone. 

Our main goal, in what follows, is to prove that $\mathcal{A} + B$ is wellposed and moreover $\mathcal{A} + B$ is bounded (that is, takes bounded sets in bounded sets).

For every sequence $(\cdot)_n \subset \mathbb{R}$ such that $x_n \to 0$ when $n \to \infty$.

2.2. **Well posedness of regular solutions.** Let us consider initially problem (1.1) with initial data in $y_0 \in D(A)$. Observe that $(\mathcal{A}, D(\mathcal{A}))$ is a maximal monotone operator in $L^2(\mathcal{M})$. Indeed, $\mathcal{A}$ is skew-adjoint if and only if $i\mathcal{A}$ is self-adjoint. Note that $i\mathcal{A} = \Delta$, soon $i\mathcal{A}$ is self-adjoint, consequently $\mathcal{A}$ is skew-adjoint and, therefore, $\mathcal{A}$ is maximal monotone.

In the sequel we shall prove that $\mathcal{B}$ is well defined. Indeed, we have

\[
\int_{\mathcal{M}} |a(x)g(y(x))|^2 \, dx \leq ||a||^2_{L^\infty(\mathcal{M})} \int_{\mathcal{M}} |g(y(x))|^2 \, dx.
\]

And, if $|g(x)| \leq 1$, since $g$ is continuous, we have that $|g(y(x))|^2 \leq K^2$. If $|g(x)| \geq 1$, taking item (b), assumption $H_2$, into account, we have $|g(y(x))|^2 \leq C^2 |y(x)|^2$. Consequently

\[
|g(y(x))|^2 \leq K^2 + C^2 |y(x)|^2, \quad \forall x \in \mathcal{M}.
\]

Substituting (4) in (3), from the compactness of $\mathcal{M}$ it results that

\[
\int_{\mathcal{M}} |a(x)g(y(x))|^2 \, dx \leq ||a||^2_{L^\infty(\mathcal{M})} \int_{\mathcal{M}} |g(y(x))|^2 \, dx \\
\leq ||a||^2_{L^\infty(\mathcal{M})} \int_{\mathcal{M}} K^2 \, dx + C^2 ||a||^2_{L^\infty(\mathcal{M})} \|y\|^2_{L^2(\mathcal{M})} \\
\leq ||a||^2_{L^\infty(\mathcal{M})} \int_{\mathcal{M}} K^2 \text{med}(\mathcal{M}) + C^2 ||a||^2_{L^\infty(\mathcal{M})} \|y\|^2_{L^2(\mathcal{M})} < \infty,
\]

which proves that $\mathcal{B}$ is wellposed and moreover $\mathcal{B}$ is bounded (that is, takes bounded sets in bounded sets).

Our main goal, in what follows, is to prove that $\mathcal{A} + \mathcal{B}$ is maximal monotone.

Initially we will prove that $\mathcal{B}$ is hemicontinuous. Indeed, we need to prove that $\forall u, v, w \in L^2(\mathcal{M}),$

\[
\lim_{t \to 0} (B(u + tv), w)_{L^2(\mathcal{M})} = (Bu, w)_{L^2(\mathcal{M})},
\]

or equivalently,

\[
\lim_{n \to \infty} (B(u + x_nv), w)_{L^2(\mathcal{M})} = (Bu, w)_{L^2(\mathcal{M})},
\]

for every sequence $(x_n)_n \subset \mathbb{R}$ such that $x_n \to 0$ when $n \to \infty$. 
Let $f_n := ag(u + x_nv)\bar{w}$, $n \in \mathbb{N}$, thus $(f_n)_n \subset L^1(\mathcal{M})$. In fact,
\[
|f_n(x)| = |a(x)g(u(x) + x_nv(x))||\bar{w}(x)|
\leq |a(x)||k + c|u(x) + x_nv(x)||\bar{w}(x)|
\leq k|a(x)||\bar{w}(x)| + c|a(x)||u(x)||\bar{w}(x)| + c|a(x)||x_n||v(x)||\bar{w}(x)|
\]
a. e. in $\mathcal{M}$.

Since $a \in L^\infty(\mathcal{M})$ and $|x_n| \leq c_1 \ \forall n \in \mathbb{N}$, yields that $f_n \in L^1(\mathcal{M})$, $\forall n \in \mathbb{N}$.

In addition, if $g(x) = k|a(x)||\bar{w}(x)| + c|a(x)||u(x)||\bar{w}(x)| + c_2|a(x)||v(x)||\bar{w}(x)|$, where $c_2 = c_1c$, then $g \in L^1(\mathcal{M})$ e $|f_n(x)| \leq g(x)$ a. e. in $\mathcal{M}$.

In the sequel, observe that
\[
\lim_{n \to \infty} a(x)g(u(x) + x_nv(x))\bar{w}(x) = a(x)g(u(x))\bar{w}(x),
\]
a. e. in $\mathcal{M}$, since $g$ is continuous. Consequently, making use of Lebesgue’s dominated convergence theorem we deduce that
\[
\int_\mathcal{M} |a(x)g(u(x) + x_nv(x))\bar{w}(x) - a(x)g(u(x))\bar{w}(x)|dx \to 0.
\]

Thus,
\[
\left| \int_\mathcal{M} [a(x)g(u(x) + x_nv(x))\bar{w}(x) - a(x)g(u(x))\bar{w}(x)]dx \right| \to 0.
\]
and, consequently,
\[
Re \int_\mathcal{M} [a(x)g(u(x) + x_nv(x))\bar{w}(x)dx \to Re \int_\mathcal{M} a(x)g(u(x))\bar{w}(x)]dx.
\]

which proves (2.6) and, consequently the hemicontinuity of $B$.

Our next step is to show that $B$ is monotone. Indeed, given $y \in L^2(\mathcal{M})$, we infer
\[
(By, y)_{L^2(\mathcal{M})} = Re \int_\mathcal{M} a(x)g(y)\overline{y}dx
= \int_\mathcal{M} a(x) \left( Re[g(y)\overline{y}] \right)_{\geq 0, \text{ from } H_1(iii)} dx \geq 0,
\]
which proves the desired.

Summarizing what we have done up to the present moment: we proved that $A$ is maximal monotone and $B$ is monotone, hemicontinuos and bounded. Then, according to Barbu [[1], corol. 1.1, pp. 39], it results that $A + B$ is maximal monotone. Then, problem (1.1) can be written as a Cauchy problem (2.1). If $u_0 \in D(A + B) = H_0^1(\mathcal{M}) \cap H^2(\mathcal{M})$, then according to Brézis [[4], Theor. 3.1, pg. 54] there exists a unique solution $y$ to problem (2.1) such that $y(t) \in D(A + B)$, $\forall \ t \geq 0$, and
\[
y \in C([0, +\infty); L^2(\mathcal{M})); \ y_t \in L^\infty(0, +\infty; L^2(\mathcal{M})).
\]

The identity of the energy (1.11) follows directly from the above class obtained by multiplying equation (1.1) by $y_t$ and integrating by parts.
2.3. Existence and Uniqueness of weak solutions. Let us consider the following o
problem:
\[
\begin{align*}
\frac{\partial}{\partial t} y &= Ty + Sy; & 0 < t < \infty \\
y(0) &= y_0
\end{align*}
\]
(2.7)
where \(T : D(T) \subset X \to X\) is a dissipative operator and \(S : D(S) \subset X \to X\) is a continuos
and dissipative operator being \(X\) a Banach space. We shall make use of the result due to
Barbu, [[1], Teo. 3.1, pp. 152]. In the present case we define \(T : D(A + B) \subset L^2(M) \to L^2(M)\)
\[S = 0 : L^2(M) \to L^2(M),\]
where \(A\) and \(B\) were defined in the previous section. We have already prove that \(T = A + B\) is maximal monotone, so that \(-T\) is m-dissipative. Obviously \(S\) is continuous and
dissipative. Consequently, problem (1.1) can be written in the following form:
\[
\begin{align*}
\frac{\partial}{\partial t} y &= -Ty + Sy; & 0 < t < \infty \\
y(0) &= y_0
\end{align*}
\]
or, in other words, the Cauchy problem (2.1) satisfies the assumptions of the precedent
result due to Barbu [1]. Thus, for all \(y_0 \in D(A + B) = H^1_0(M) \cap H^2(M) = L^2(M)\),
there exists a unique \(y \in C([0, +\infty); L^2(M))\) which is a generalized solution to problem
(2.1).

The identity of the energy (1.11) follows from standard density arguments.

2.4. Uniform Decay Rates. Let us consider the problems:
\[
\begin{align*}
i \partial_t y + \Delta y + ia(x) g(y) &= 0 & \text{in } M \times (0, \infty) \\
y &= 0 & \text{on } \partial M \times (0, \infty) \\
y(0) &= y_0 & \text{in } M,
\end{align*}
\]
(2.8)
and
\[
\begin{align*}
i \partial_t \varphi + \Delta \varphi &= 0 & \text{in } M \times (0, \infty) \\
\varphi &= 0 & \text{on } \partial M \times (0, \infty) \\
\varphi(0) &= y_0 & \text{in } M,
\end{align*}
\]
(2.9)
Let us assume that observability inequality (1.4) holds
\[
||y_0||^2_{L^2(M)} \leq c \int_0^T \int_\omega |\varphi|^2 \, dx \, dt, \quad \text{for all } T > T_0
\]
(2.11)
where \(\omega \subset M\) is an open set contained properly in \(M\) and the function \(a(\cdot)\) satisfies the
assumption (1.3).
We observe that \( y = \varphi + v \) is the solution of (2.8), where \( \varphi \) is the solution of (2.9) and \( v \) is the solution of (2.10). In addition, \( y = \varphi + v = 0 \) on \( \partial \mathcal{M} \times (0, \infty) \) and \( y(0) = \varphi(0) + v(0) = y_0 \).

Multiplying (2.8) by \( i \) and integrating over \( \mathcal{M} \), we have:
\[
\int_{\mathcal{M}} \partial_t y \overline{y} \, dx - i \int_{\mathcal{M}} |\nabla y|^2 \, dx + \int_{\mathcal{M}} a(x) g(y) \overline{y} \, dx = 0.
\]

Taking the real part and making use of \((H_1)\), item (iii), we deduce that
\[
\text{Re} \int_{\mathcal{M}} \partial_t y \overline{y} \, dx + \text{Re} \int_{\mathcal{M}} a(x) g(y) \overline{y} \, dx = 0,
\]
or, in other words,
\[
\frac{1}{2} \frac{d}{dt} \|y(t)\|_{L^2(\mathcal{M})}^2 = -\text{Re} \int_{\mathcal{M}} a(x) g(y) \overline{y} \, dx \leq 0.
\]

Having in mind that the energy is given by (2.2), we infer
\[
\frac{d}{dt} E(t) = \frac{d}{dt} \|y(t)\|_{L^2(\mathcal{M})}^2 \leq 0,
\]
which tells us that \( E(t) \) is a non-increasing function in the parameter \( t \).

We shall prove that
\[
\frac{1}{2} E(T) \leq c(T) \int_0^T \int_{\mathcal{M}} a(x) \left( |y|^2 + |g(y)|^2 \right) \, dx \, dt \tag{2.12}
\]
for all \( T > T_0 > 0 \), for some \( T_0 \).

Since the energy is non-increasing, from (2.11) we deduce that
\[
\frac{1}{2} E(T) \leq \frac{1}{2} E(0) = \frac{1}{2} \|y_0\|_{L^2(\mathcal{M})}^2 = c \int_0^T \int_{\omega} |\varphi|^2 \, dx \, dt
\]
\[
\leq c_1 \int_0^T \int_{\omega} a(x) |y - v|^2 \, dx \, dt
\]
\[
\leq c_2 \int_0^T \int_{\omega} a(x) \left( |y|^2 + |v|^2 \right) \, dx \, dt
\]
\[
\leq c_2 \int_0^T \int_{\mathcal{M}} a(x) \left( |y|^2 + |g(y)|^2 \right) \, dx \, dt + c_2 \int_0^T \int_{\mathcal{M}} a(x) |v|^2 \, dx \, dt \tag{\lor}
\]
\]

Estimative for \( I \).

We define the following linear and continuous
\[
T : L^2(\mathcal{M}) \times L^1(0,T;L^2(\mathcal{M})) \rightarrow L^\infty(0,T;L^2(\mathcal{M}))
\]
\[
(z_0, f) \mapsto T(z_0, f) = z
\]
where $z$ is the solution of the problem

\[
\begin{aligned}
&\begin{cases}
  i \partial_t z + \Delta z = f & \text{in } \mathcal{M} \times (0, \infty) \\
  z = 0 & \text{on } \partial \mathcal{M} \times (0, \infty) \\
  z(0) = z_0 \in L^2(\mathcal{M}),
\end{cases}
\end{aligned}
\]

(2.14)

Clearly $T$ is linear. We shall prove that $T$ is continuous. Indeed, since $z$ is solution of (2.14), we have that $z$ satisfies the integral equation

\[
z(t) = S(t)z_0 + \int_0^t S(t-s) f(s) \, ds
\]

where $S(t)$ is the semigroup generated by the maximal monotone operator $A$. Thus, having in mind that $||S(t)||_{L^2(\mathcal{M})} \leq M$, we infer

\[
||z(t)||^2_{L^2(\mathcal{M})} = \left|\left| S(t)z_0 + \int_0^t S(t-s) f(s) \, ds \right|\right|_{L^2(\mathcal{M})}^2
\]

\[
\leq c ||S(t)z_0||^2_{L^2(\mathcal{M})} + c \left|\left| \int_0^t S(t-s) f(s) \, ds \right|\right|_{L^2(\mathcal{M})}^2
\]

\[
\leq c_1 ||z_0||^2_{L^2(\mathcal{M})} + c_2 \left( \int_0^t ||f(s)||^2_{L^2(\mathcal{M})} \, ds \right)^2
\]

\[
\leq C(||z_0||^2_{L^2(\mathcal{M})} + ||f||_{L^1(0,T;L^2(\mathcal{M}))}^2) = C(||z_0,f||^2_{L^2(\mathcal{M}) \times L^1(0,T;L^2(\mathcal{M}))}),
\]

which proves the desired.

In this way, once $L^\infty(0,T;L^2(\mathcal{M})) \hookrightarrow L^2(0,T:L^2(\mathcal{M}))$ and setting $f(t) = a(x)g(y(t))$, $z_0 = 0$, we have from Hölder inequality and the continuity of $T$ that

\[
I := \int_0^T \int_\mathcal{M} a(x)|v|^2 \, dx \, dt
\]

(2.15)

\[
\leq ||a||_{L^\infty(\mathcal{M})} \left|\left| ||v||^2_{L^2(0,T;L^2(\mathcal{M}))} \right|\right|
\]

\[
\leq ||a||_{L^\infty(\mathcal{M})} \left|\left| ||v||^2_{L^\infty(0,T;L^2(\mathcal{M}))} \right|\right|
\]

\[
\leq ||a||_{L^\infty(\mathcal{M})} \left|\left| ||a(x)g(y)||^2_{L^1(0,T;L^2(\mathcal{M}))} \right|\right|
\]

\[
\leq c_3 \left( \int_0^T \left[ \int_\mathcal{M} a(x) \left|g(y)\right|^2 \, dx \right]^{\frac{1}{2}} \right)^2
\]

\[
\leq T^2 c_3 \int_0^T \int_\mathcal{M} a(x) \left|g(y)\right|^2 \, dx
\]

\[
\leq c_4(T) \int_0^T \int_\mathcal{M} [a(x) \left|g(y)\right|^2 + a(x) \left|y\right|^2] \, dx.
\]

Finally, combining (2.15) and (2.13) we obtain the desired estimate of energy given in (2.12).
3. Plate Equation

In this section we shall study the problem (1.2).

3.1. Notation. Let us consider the space \( L^2(\mathcal{M}) \) of complex valued functions on \( \mathcal{M} \). It is a real Hilbert space when endowed with the inner product

\[
(y, z)_{L^2(\mathcal{M})} = \Re \int_{\mathcal{M}} y(x) \overline{z(x)} \, dx.
\]

We shall denote \( V = H^1_0(\mathcal{M}) \cap H^2(\mathcal{M}) \) with the corresponding norm in \( V \) as

\[
|||y|||_V^2 = ||\Delta y||^2_{L^2(\mathcal{M})},
\]

which is equivalent to the norm in \( H^2(\mathcal{M}) \).

Let \( Y = \begin{pmatrix} y \\ z \end{pmatrix} \), so we shall write problem below as a Cauchy problem in \( L^2(\mathcal{M}) \) of the form:

\[
\begin{aligned}
\partial_t Y + AY + FY &= 0 \\
Y(0) &= Y_0,
\end{aligned}
\]

(3.16)

where \( A : D(A) \subset \mathcal{H} \to \mathcal{H}, D(A) = \{ u \in H^1_0(\mathcal{M}) \cap H^4(\mathcal{M}); \Delta u = 0 \text{ on } \partial \mathcal{M} \} \times V \), the Hilbert space \( \mathcal{H} \) is defined by \( V \times L^2(\mathcal{M}) \) and the operator \( A \) is defined by

\[
A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ \Delta^2 y \end{pmatrix}, \quad \begin{pmatrix} y \\ z \end{pmatrix} \in D(A).
\]

Note that the inner product and the norm in \( D(A) \) and \( H \) are respective

\[
|||Y|||_{D(A)}^2 = ||y||^2_V + ||z||^2_{L^2(\mathcal{M})} + ||z||^2_{L^2(\mathcal{M})}; \quad ||Y||^2_{\mathcal{H}} = ||y||^2_V + ||z||^2_{L^2(\mathcal{M})}.
\]

The operator \( F : \mathcal{H} \to \mathcal{H} \) is defined by

\[
F \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ a(\cdot)g(z) \end{pmatrix}.
\]

We shall prove that \( A + F \) is a maximal monotone operator so that problem (1.2) is wellposed and, in addition, the energy is given by

\[
E(t) := \frac{1}{2} \int_{\mathcal{M}} \left[ |y_t(x,t)|^2 + |\Delta y(x,t)|^2 \right] \, dx,
\]

(3.17)

satisfies the identity of the energy

\[
E(t) + 2 \int_t^s g(y) \overline{y_t} \, dx dt = E(s), \quad \text{where then } E(t) \leq E(s), \; t \geq s \geq 0.
\]

(3.18)
3.2. **Well Posedness of regular solutions.** Let us consider initially problem (1.2) and let us make the change of variables \( Y = \begin{pmatrix} y \\ y_t \end{pmatrix} \). Thus,

\[
\frac{dY}{dt} = \begin{pmatrix} y_t \\ y_{tt} \end{pmatrix} = \begin{pmatrix} -\Delta^2 y - a(x)g(y_t) \\ -\Delta^2 y \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\Delta^2 y - a(x)g(y_t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\Delta^2 y \\ 0 \\ 0 \end{pmatrix} = : -A + F
\]

obtaining the Cauchy problem (3.16).

Note that \( A \) is monotone since

\[
\langle A \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle = \langle -z\Delta^2 y, z \rangle_{L^2(M)} + \text{Re} \int_M \Delta^2 y \overline{v} \, dx = \text{Re} \int_M -\Delta z \Delta y \, dx + \text{Re} \int_M \Delta y \overline{\Delta z} \, dx = 0,
\]

where we have used the fact that \( \text{Re}(-z_1 \overline{z_2}) + \text{Re}(\overline{z_1} z_2) = 0 \).

Using Lax-Milgram’s Lemma, we also show that \( A \) is maximal.

In the sequel we shall prove that \( F \) is well defined, it is monotone, hemicontinuous and bounded from \( \mathcal{H} \) in \( H \). Note that from the computations we have done for the Schrödinger equation, we guarantee such mentioned properties for the operator \( F \), once \( F \approx B \) where \( B \) is defined by:

\[
B : L^2(M) \to L^2(M), \quad z \mapsto a(x)g(z)
\]

Then, according to Barbu [1], corol. 1.1, pp. 39, it results that \( A + F =: D(A + F) \subset \mathcal{H} \to \mathcal{H} \) is maximal monotone (note that \( D(A + F) = \{ u \in H^1_0(M) \cap H^4(M); \Delta u = 0 \text{ on } \partial M \} \times V \).

Consequently, problem (1.2) can be writen as the Cauchy problem (3.16). If \( Y_0 \in D(A + F) \), that is, \( y_0 \in \{ u \in H^1_0(M) \cap H^4(M); \Delta u = 0 \text{ on } \partial M \} \) and \( y_1 \in V \), then from Brézis [4], Theor. 3.1, pg. 54] there exists a unique solution \( Y \) of (2.1) such that \( Y(t) \in D(A + F), \forall \, t \geq 0 \), and

\[
Y \in C([0, +\infty); \mathcal{H}); \ Y_t \in L^\infty(0, +\infty; \mathcal{H}).
\]
In other words,
\[ y \in C([0, +\infty); V) \cap C^1([0, +\infty); L^2(\mathcal{M})) \]
\[ y_t \in L^\infty(0, +\infty; V) \cap W^{1,\infty}(0, +\infty; L^2(\mathcal{M})). \]

3.3. **Existence and Uniqueness of weak solution.** Let us consider the following problem:
\[
\begin{cases}
\partial_t y = Ty + Sy; & 0 < t < \infty \\
y(0) = y_0
\end{cases}
\] (3.19)
where \( T : D(S) \subset H \to H \) is a m-dissipative operator, \( S : D(S) \subset H \to H \) is continuous and dissipative and \( H \) is a Banach space. We shall make use of a result due to Barbu, [1], Teo. 3.1, pp. 152.

In our particular case, let us define
\[ T : D(A + F) \subset \mathcal{H} \to \mathcal{H} \]
\[ S = 0 : \mathcal{H} \to \mathcal{H} \]
where \( A \) and \( F \) where defined in the previous subsection. We have already seen that \( T = A + F \) is maximal monotone, so that \(-T\) is m-dissipative. Evidently \( S \) is continuous and dissipative.

Consequently, problem (3.16) can be rewritten of the following form:
\[
\begin{cases}
\partial_t Y = -TY + SY; & 0 < t < \infty \\
Y(0) = Y_0
\end{cases}
\]
or still, like in (3.16) and satisfies the assumptions of the previous result mentioned in Barbu [1]. Thus, for all \( Y_0 \in D(A + F) = \mathcal{H} \), there exists a unique \( Y \in C([0, +\infty); \mathcal{H}) \) solution of (3.16). So, there exists a unique \( y \in C([0, +\infty); V) \cap C^1((0, +\infty); L^2(\mathcal{M})) \) solution of (1.2).

3.4. **Uniform Decay Rates.** Let us consider the problems:
\[
\begin{cases}
\partial_{tt} y + \Delta^2 y + a(x) g(\partial_t y) = 0 & \text{in } \mathcal{M} \times (0, \infty) \\
y = \Delta y = 0 & \text{on } \partial\mathcal{M} \times (0, \infty) \\
y(0) = y_0 & \text{in } \mathcal{M},
\end{cases}
\] (3.20)
\[
\begin{cases}
\partial_{tt} \varphi + \Delta^2 \varphi = 0 & \text{in } \mathcal{M} \times (0, \infty) \\
\varphi = \Delta \varphi = 0 & \text{on } \partial\mathcal{M} \times (0, \infty) \\
\varphi(0) = y_0 & \text{in } \mathcal{M},
\end{cases}
\] (3.21)
and
\[
\begin{cases}
\partial_{tt} v + \Delta^2 v = -a(x) g(\partial_t y) & \text{in } \mathcal{M} \times (0, \infty) \\
v = \Delta v = 0 & \text{on } \partial\mathcal{M} \times (0, \infty) \\
v(0) = 0 & \text{in } \mathcal{M},
\end{cases}
\] (3.22)
We shall assume that the observability inequality estimate holds:

\[
\int_{\mathcal{M}} \left[ |\varphi_t(0)|^2 + |\Delta \varphi(0)|^2 \right] \ dx \leq c \int_{0}^{T} \int_{\omega} |\varphi_t|^2 \ dx \ dt, \quad \text{for all} \ T > T_0
\]

where \( \omega \subset \mathcal{M} \) is an open set contained properly in \( \mathcal{M} \) and the function \( a(\cdot) \) satisfies (1.3).

Note that \( y = \varphi + v \) is the solution of (3.20), where \( \varphi \) is the solution of (3.21) and \( v \) is the solution of (3.22). In addition, \( y = \varphi + v = 0 \) in \( \partial \mathcal{M} \times (0, \infty) \) and \( y(0) = \varphi(0) + v(0) = y_0 \).

Multiplying (3.20) by \( y_t \) and integrating over \( \mathcal{M} \), we infer

\[
\int_{\mathcal{M}} y_{tt} y_t \ dx + \int_{\mathcal{M}} \Delta^2 y \ y_t \ dx + \int_{\mathcal{M}} a(x) \ g(y_t) y_t \ dx = 0.
\]

Taking the real part and making use of \((H_1)\), item (iii), we have

\[
\text{Re} \int_{\mathcal{M}} y_{tt} y_t \ dx + \text{Re} \int_{\mathcal{M}} \Delta y \ \Delta y_t \ dx + \int_{\mathcal{M}} a(x) \ Re\{g(y_t) y_t\} \ dx \geq 0.
\]

In other words,

\[
\frac{1}{2} \frac{d}{dt} \|y_t\|_{L^2(\mathcal{M})}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta y\|_{L^2(\mathcal{M})}^2 \Delta y = -\int_{\mathcal{M}} a(x) \ Re\{g(y_t) y_t\} \ dx \leq 0.
\]

From the identity of the energy given in (3.18), we deduce

\[
\frac{d}{dt} E(t) = \frac{1}{2} \frac{d}{dt} \|y_t(t)\|_{L^2(\mathcal{M})}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta y(t)\|_{L^2(\mathcal{M})}^2 \leq 0,
\]

which implies that \( E(t) \) is non-increasing on the parameter \( t \).

We would like to show that

\[
\frac{1}{2} E(T) \leq c(T) \int_{0}^{T} \int_{\mathcal{M}} a(x) [ \|y_t\|^2 + |g(y_t)|^2 ] \ dx \ dt
\]

for all \( T > T_0 \).

Since the energy is non-increasing, from (3.23) we deduce

\[
\frac{1}{2} E(T) \leq \frac{1}{2} E(0)
\]

\[
= \frac{1}{2} \int_{\mathcal{M}} \left[ |y_t(0)|^2 + |\Delta y(0)|^2 \right] \ dx
\]

\[
\leq c \int_{0}^{T} \int_{\omega} |\varphi_t|^2 \ dx \ dt
\]

\[
\leq c_1 \int_{0}^{T} \int_{\omega} a(x) |y_t - v_t|^2 \ dx \ dt
\]

\[
\leq c_2 \int_{0}^{T} \int_{\omega} a(x) \left( |y_t|^2 + |v_t|^2 \right) \ dx \ dt
\]

\[
\leq c_2 \int_{\mathcal{M}} a(x) [ |y_t|^2 + |g(y_t)|^2 ] \ dx \ dt + c_2 \int_{0}^{T} \int_{\mathcal{M}} a(x) |v_t|^2 \ dx \ dt \quad \text{:=} I
\]

Estimative for \( I \).
We define the following linear and continuous operator

\[ T : \mathcal{H} \times L^1(0, T; L^2(\mathcal{M})) \to L^\infty(0, T; L^2(\mathcal{M})) \]

\[ ((z_0, z_1), f) \mapsto T(z_0, f) = z_t \]

where \( z \) is the solution to problem

\[
\begin{aligned}
z_{tt} + \Delta^2 z &= f \quad \text{in } \mathcal{M} \times (0, \infty) \\
z_t &= \Delta z = 0 \quad \text{on } \partial \mathcal{M} \times (0, \infty) \\
z(0) &= z_0 \in \mathcal{V}; \quad z_t(0) = z_1 \in L^2(\mathcal{M}),
\end{aligned}
\]

(3.26)

Clearly \( T \) is linear. It remains to prove the continuity of \( T \). Indeed, once \( Z = \left( \begin{array}{c} z \\ z_t \end{array} \right) \) is the solution of problem

\[
\begin{aligned}
Z_t + AZ &= F \\
Z(0) &= Z_0
\end{aligned}
\]

where \( A = \left( \begin{array}{cc} 0 & -I \\ \Delta^2 & 0 \end{array} \right) \) and \( F = \left( \begin{array}{c} 0 \\ f \end{array} \right) \), we have that \( Z \) satisfies the integral equation

\[ Z(t) = S(t)Z_0 + \int_0^t S(t-s)F(s)\, ds \]

where \( S(t) \) is the semigroup generated by the maximal monotone operator \( A \). Thus, having in mind that \( ||S(t)||_{\mathcal{L}(\mathcal{H})} \leq M \), we infer

\[
||z_t(t)||_{L^2(\mathcal{M})}^2 \leq ||Z(t)||_{\mathcal{H}}
\]

\[
\leq c||S(t)||_{\mathcal{L}(\mathcal{H})}^2 ||(z_0, z_1)||_{\mathcal{H}}^2 + c \left| \left| \int_0^t S(t-s) \left( \begin{array}{c} 0 \\ f(s) \end{array} \right) \right| \right|_{\mathcal{H}}^2 ds \leq c_1 ||(z_0, z_1)||_{\mathcal{H}}^2 + c_2 \left( \int_0^t \left| \left( \begin{array}{c} 0 \\ f(s) \end{array} \right) \right|_{\mathcal{H}}^2 \right)^2 ds \leq c_3 ||(z_0, z_1)||_{\mathcal{H}}^2 + c_3 ||f||_{L^1(0,T;L^2(\mathcal{M}))}^2 = c_4 ||(z_0, f)||_{\mathcal{H} \times L^1(0,T;L^2(\mathcal{M}))}^2
\]

Therefore,

\[ ||z_t||_{L^\infty(0, T; L^2(\mathcal{M}))} \leq c_4 ||(z_0, f)||_{\mathcal{H} \times L^1(0,T;L^2(\mathcal{M}))}^2, \]

where \( c_4 = \max \{c_1, c_3\} \), which proves the continuity of \( T \).
In this way, since \( L^2(0, T; L^2(\mathcal{M})) \) and assuming that \( f(t) = a(x) g(y(t)) \), \( z_0 = z_1 = 0 \), it holds that \( f \in L^1(0, T; L^2(\mathcal{M})) \), since, from Hölder inequality,

\[
\int_0^T ||a(\cdot) g(y_t)||_{L^2(\mathcal{M})} \, dt = \int_0^T \left( \int_{\mathcal{M}} |a(x)|^2 |g(y_t(x, t)|^2 \, dx \right)^{\frac{1}{2}} \, dt \\
\leq ||a||_{L^\infty(\mathcal{M})} \int_0^T \left( \int_{\mathcal{M}} |g(y_t(x, t)|^2 \, dx \right)^{\frac{1}{2}} \, dt \\
\leq ||a||_{L^\infty(\mathcal{M})} T^{\frac{1}{2}} \int_0^T \left( \int_{\mathcal{M}} [k^2 + c_3 |y_t|^2] \, dx \right)^{\frac{1}{2}} \, dt \\
\leq c_4 \left( k^2 \text{med}(\mathcal{M}) T + c_3 \int_0^T \int_{\mathcal{M}} |y_t|^2 \, dx \, dt \right)^{\frac{1}{2}} \\
\leq c_4 (c_5(T) + c_6 ||y_t||_{L^2(0, T; L^2(\mathcal{M}))})^{\frac{1}{2}} < +\infty,
\]

since that \( y_t \in C([0, +\infty); L^2(\mathcal{M})) \), thus \( y_t \in L^\infty(0, T; L^2(\mathcal{M})) \) \( \hookrightarrow L^2(0, T; L^2(\mathcal{M})). \)

Then, from the continuity of \( T \) and, again, from Hölder inequality we deduce that

\[
(3.27) \quad I := c_2 \int_0^T \int_\omega |a(x)| |v_t|^2 \, dx \, dt \\
\leq c_2 \int_0^T \int_{\mathcal{M}} |a(x)| |v_t|^2 \, dx \, dt \\
\leq c_2 ||a||_{L^\infty(\mathcal{M})} ||v_t||_{L^2(0, T; L^2(\mathcal{M}))}^2 \\
\leq c_3 ||a||_{L^\infty(\mathcal{M})} ||v_t||_{L^\infty(0, T; L^2(\mathcal{M}))}^2 \\
\leq c_4 ||a||_{L^\infty(\mathcal{M})} ||a(x) g(y_t)||_{L^1(0, T; L^2(\mathcal{M}))} \\
\leq c_5 \left( \int_0^T \left[ \int_{\mathcal{M}} |a(x)| |g(y_t)|^2 \, dx \right]^{\frac{1}{2}} \right)^2 \\
\leq c_6(T) \int_0^T \int_{\mathcal{M}} |a(x)| |g(y)|^2 \, dx \\
\leq c_6(T) \int_0^T \int_{\mathcal{M}} [a(x) |g(y_t)|^2 + a(x) |y_t|^2] \, dx.
\]

Finally, combining (3.27) and (3.25) we obtain the desired estimate of energy given (3.24).

4. Exemples

Let us present some nice examples, different of the exponential decay, that can be found in Lasiecka and Tirioggianni [25] that we have borrowed.

- Example 1: Case \( g(z) = |z|^r z \) near the origin, \( r > 0 \).
In this case the decay rate is given by

\[ E(t) \leq C(E(0)) \left[ \left( \frac{E(0)}{4} \right)^{-\frac{\tau}{2}} + \frac{r}{3} (t - T) \right]^{-\frac{\tau}{2}}, \text{ for } t \geq T. \]

- Example 2: Case \( g(z) = |z|^2 e^{-1/|z|^2} \), near the origin.
  In this case the decay is given by
  \[ E(t) \leq C(E(0)) \left[ \ln \left( e^{\frac{4}{E(0)}} + \frac{4}{3} (t - T) \right) \right]^{-1}, \text{ for } t \geq T. \]

- Example 3: Case \( g(z) = \frac{1}{|z|^r} z, 0 < r < 1 \), near the origin.
  The decay rate, in this case is
  \[ E(t) \leq C(E(0)) \left[ \left( \frac{4}{E(0)} \right)^{m-1} + \frac{m-1}{4} (t - T) \right]^{1-m}, \quad m = \frac{1}{2} \left( 1 + \frac{1}{1-r} \right), \text{ for } t \geq T. \]

5. The Wave Equation - Multipliers versus Microlocal Analysis

In this section we would like to present, for the wave equation, a comparison between the sharpness of the microlocal analysis against the technique obtained by the multipliers. So, let us consider the linear wave equation subject to a nonlinear and localized dissipative term given by

\[
\begin{cases}
  u_{tt} - \Delta u + a(x) g(u_t) = 0 & \text{in } M \times (0, \infty], \\
  u = 0 & \text{on } \partial M \times [0, \infty], \\
  u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x) & \text{in } M.
\end{cases}
\]

It is well-known that when \( g \equiv 0 \), the geometric control condition (GCC) is equivalent to the observability inequality. Consequently, the method presented in the previous section "generalizes" (in a certain sense) the results due to Cavalcanti et. al. [9], [10] regarding the optimal choice of dissipative regions. Indeed, the results in the works [9], [10] are obtained by using multipliers of the first order. It has been proved in Miller [34] that the use of multipliers of the first order will never reach the sharpness of the microlocal analysis. In order to make easier this comparison in the present context, let us consider the wave propagation on regular compact surfaces without boundary.

It is well-known that the multipliers work nicely with the semi-linear wave equation:

\[ u_{tt} - \nabla u + f(u) + a(x) g(u_t) = 0 \text{ in } M \times (0, \infty). \]

So, multipliers combined with integral estimates of energy (as in Lasiecka and Tataru [20]) plus a nice unique continuation property (as in Triggiani and Yao [41]) are the main ingredients to derive general decay rate estimates of the energy avoiding put damping in strategic regions on the surface as follows:
5.1. **External Vision - Radial Multipliers.** We observe that in the particular case when \( m(x) = x - x^0, x \in \mathbb{R}^3 \) and \( x^0 \in \mathbb{R}^3 \) is a fixed point in \( \mathbb{R}^3 \), we have

\[
\text{div} m = 3, \quad \text{div}_T m_T = 2 + (m \cdot \nu)Tr B.
\]

where \( B \) is the second fundamental form of \( M \) (the shape operator) and \( Tr \) is the trace. Let \( \varphi : M \rightarrow \mathbb{R}^3 \) be a smooth function and \( m \) the radial vector field defined above. We also have,

\[
\nabla_T \varphi \cdot \nabla_T m_T \cdot \nabla_T \varphi = |\nabla_T \varphi|^2 + (m \cdot \nu)(\nabla_T \varphi \cdot B \cdot \nabla_T \varphi).
\]

5.1.1. **Shape Operator.** The sign of \( B \) can change in the literature. In our case, we remember that \( B = -dN \), where \( N \) is the Gauss map related to \( \nu \).

The formulas (5.28) can be rewritten by

\[
\text{div} m = 3, \quad \text{div}_T m_T = 2 + 2H (m \cdot \nu).
\]

where \( H = \frac{trB}{2} \) is the mean curvature of \( M \).

5.2. **Uniform Decay Rate Estimates.** We shall work with regular solutions and by use of density arguments we can extend our results for weak solutions.

Our main task is to obtain the following estimate:

\[
\int_0^T E(t) \, dt \leq C_1 E(T) + C_2 \int_0^T \int_M a(x) \left( g(u_t)^2 + u_t^2 \right) \, dM dt,
\]

for some positive constants \( C_i > 0, i = 1, 2 \), where \( C_1 \) can not depend on \( T \) but \( C_2 \) eventually can. From this estimate we deduce the desired decay rates estimates following (verbatim) the ideas firstly introduced by Lasiecka and Tataru [20]. From this purpose, we shall need two fundamental identities that will be proved in the sequel.

5.2.1. **The First Identity.**

**Lemma 5.1.** Let \( M \subset \mathbb{R}^3 \) be oriented regular compact surface without boundary and \( q \) a regular vector field with \( q = q_T + (q \cdot \nu)\nu \). Then, for every regular solution \( u \) of (5) we have the following identity

\[
\left[ \int_M u_t q_T \cdot \nabla_T u \, dM \right]^T_0 \\
+ \frac{1}{2} \int_0^T \int_M \left( \text{div}_T q_T \right) \left\{ |u_t|^2 - |\nabla_T u|^2 \right\} \, dM dt \\
+ \int_0^T \int_M \nabla_T u \cdot \nabla_T q_T \cdot \nabla_T u \, dM dt \\
+ \int_0^T \int_M a(x) g(u_t) (q_T \cdot \nabla_T u) \, dM dt = 0.
\]
Employing (5.31) with \( q(x) = m(x) = x - x^0 \) for some \( x^0 \in \mathbb{R}^3 \) fixed and taking (5.28) and (5.29) into account, we infer

\[
\left[ \int_{\mathcal{M}} u_t m_T \cdot \nabla_T u \, d\mathcal{M} \right]_0^T + \int_0^T \int_{\mathcal{M}} \left\{ |u_t|^2 - |\nabla_T u|^2 \right\} \, d\mathcal{M} \, dt + \int_0^T \int_{\mathcal{M}} |\nabla_T u|^2 + (m \cdot \nu)(\nabla_T u \cdot B \cdot \nabla_T u) \, d\mathcal{M} \, dt + \int_0^T \int_{\mathcal{M}} (m \cdot \nu)H \left\{ |u_t|^2 - |\nabla_T u|^2 \right\} \, d\mathcal{M} \, dt + \int_0^T \int_{\mathcal{M}} a(x)g(u_t)(m_T \cdot \nabla_T u) \, d\mathcal{M} \, dt = 0. \tag{5.32}
\]

5.2.2. Second Identity.

**Lemma 5.2.** Let \( u \) be a weak solution to problem (5) and \( \xi \in C^1(\mathcal{M}) \). Then

\[
\left[ \int_{\mathcal{M}} u_t \xi u \, d\mathcal{M} \right]_0^T = \int_0^T \int_{\mathcal{M}} \xi |u_t|^2 \, d\mathcal{M} \, dt - \int_0^T \int_{\mathcal{M}} \xi |\nabla_T u|^2 \, d\mathcal{M} \, dt - \int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot \nabla_T \xi) u \, d\mathcal{M} \, dt - \int_0^T \int_{\mathcal{M}} a(x)g(u_t)\xi u \, d\mathcal{M} \, dt. \tag{5.33}
\]

Substituting \( \xi = \frac{1}{2} \) in (5.33) and combining the obtained result with identity (5.32) we deduce

\[
\left[ \int_{\mathcal{M}} u_t m_T \cdot \nabla_T u \, d\mathcal{M} \right]_0^T + \frac{1}{2} \left[ \int_{\mathcal{M}} u_t u \, d\mathcal{M} \right]_0^T + \int_0^T E(t) \, dt + \int_0^T \int_{\mathcal{M}} a(x)g(u_t)(m_T \cdot \nabla_T u) \, d\mathcal{M} \, dt + \frac{1}{2} \int_0^T \int_{\mathcal{M}} a(x)g(u_t)u \, d\mathcal{M} \, dt = -\int_0^T \int_{\mathcal{M}} (m \cdot \nu)H \left\{ |u_t|^2 - |\nabla_T u|^2 \right\} \, d\mathcal{M} \, dt - \int_0^T \int_{\mathcal{M}} (m \cdot \nu)(\nabla_T u \cdot B \cdot \nabla_T u) \, d\mathcal{M} \, dt. \tag{5.34}
\]

Observe that some terms in (5.34) are easily handled by using the Cauchy Schwarz and Poincaré inequalities as well as the inequality \( ab \leq \frac{1}{4}a^2 + \varepsilon b^2 \) and exploiting the identity of the energy

\[ E(T) - E(0) = -\int_0^T \int_{\mathcal{M}} a(x)g(u_t)u_t \, d\mathcal{M} \, dt. \]
5.2.3. Analysis of the terms which involve the shape operator $B$. Let us focus our attention on the shape operator $B : T_xM \to T_xM$. There exists an orthonormal basis $\{e_1, e_2\}$ of $T_xM$ such that $Be_1 = k_1e_1$ and $Be_2 = k_2e_2$ and $k_1$ and $k_2$ are the principal curvatures of $M$ at $x$. The matrix of $B$ with respect to the basis $\{e_1, e_2\}$ is given by

$$
B := \begin{pmatrix}
k_1 & 0 \\
0 & k_2
\end{pmatrix}.
$$

Setting $\nabla_T u = (\xi, \eta)$ the coordinates of $\nabla_T u$ in the basis $\{e_1, e_2\}$, for each $x \in M$, we deduce that

$$
\nabla_T u \cdot B \cdot \nabla_T u = k_1\xi^2 + k_2\eta^2.
$$

(5.35)

Then, from (5.51), we infer

$$
(m \cdot \nu) \left[ (\nabla_T u \cdot B \cdot \nabla_T u) - \frac{1}{2} \text{Tr}(B)|\nabla_T u|^2 \right]
= (m \cdot \nu) \left[ \frac{k_1 - k_2}{2} \xi^2 + \frac{(k_2 - k_1)}{2} \eta^2 \right].
$$

(5.36)

**Remark:** Observe that this is the precise moment that the intrinsic properties of the manifold $M$ appear, that is,

5.2.4. Necessity of umbilical nondissipative region (by parts). We strongly need that the term $-\int_0^T \int_M (m \cdot \nu) Hu_t^2 \, dM \, dt$ lies in a region where the damping term is effective. Remember that the damping term is effective on an open set $M_\star$ which contains $M \setminus \cup_{i=1}^k M_{0i}$. So, assuming that $H \leq 0$ and since $m(x) \cdot \nu(x) \leq 0$ on $M_0$, we have

$$
-\int_0^T \int_{M_0} (m \cdot \nu) H |u_t|^2 \, dM \, dt \leq 0.
$$

In addition, supposing that $M_{0i}$ is umbilical for every $i = 1, \ldots, k$, then, having (5.36) in mind, we also have that

$$
\int_0^T \int_{M_{0i}} (m \cdot \nu) \left[ H |\nabla_T u|^2 - (\nabla_T u \cdot B \cdot \nabla_T u) \right] \, dM \, dt = 0,
$$

for $i = 1, \ldots, k$.

Observe that if $M_0$ is a peace of a conical surface $M$, that is, $m(x) \cdot \nu(x) = 0$, for all $x \in M_0$, we also deduce that

$$
-\int_0^T \int_{M_0} (m \cdot \nu) H |u_t|^2 \, dM \, dt = 0.
$$

$$
\int_0^T \int_{M_0} (m \cdot \nu) \left[ H |\nabla_T u|^2 - (\nabla_T u \cdot B \cdot \nabla_T u) \right] \, dM \, dt = 0.
$$

5.2.5. Surfaces constituted by umbilical parts: $H \leq 0, |k_1 - k_2| < \varepsilon, (x-x_0) \cdot \nu(x) \leq 0$. See picture 2

5.2.6. Surfaces constituted by conical parts: $(x-x_0) \cdot \nu(x) = 0$. See picture 3.
5.2.7. *Inverse Inequality.* Note that if \( a = 0 \), that is, if one has the linear wave equation

\[
\begin{cases}
  u_{tt} - \Delta_M u = 0 & \text{in } M \times (0, \infty) \\
u(x,0) = u_0(x); \ u_t(x,0) = u_1(x), & x \in M.
\end{cases}
\]

then, \( E(T) = E(0) \) for all \( T \geq 0 \) and from (5.34) we easily deduce the inverse inequality

\[
E_0 \leq C \int_0^T \int_{M_2} \left[ a_1^2 + \left| \nabla_T u \right|^2 \right] dM dt,
\]

where \( C \) is a positive constant and \( M_2 = M \setminus \bigcup_{i=1}^k M_{0i} \). The inverse inequality (5.37) says that if \( u \) is zero in \( \bigcup_{i=1}^k M_{0i} \), then \( u = 0 \) in the whole \( M \). In other words, this is an unique continuation principle for any finite union of disjoint and umbilical and conical sets.

Returning to (5.34), now considering, again, the damping term \( a(x)g(u_t) \) and the geometric considerations above mentioned into account, we deduce,

\[
\frac{1}{2} \int_0^T E(t) \, dt \leq |\chi| + C_1 \int_0^T \int_M \left( a(x) (g(u_t))^2 \right) dM dt + C_1 \int_0^T \int_{M_2} \left[ \left| \nabla_T u \right|^2 + a(x) u_t^2 \right] dM dt
\]
where
\[
\chi = - \left[ \int_{\mathcal{M}} u_t m T \cdot \nabla T u \, d\mathcal{M} \right]_0^T - \frac{1}{2} \left[ \int_{\mathcal{M}} u_t u \, d\mathcal{M} \right]_0^T 
\]
\[
C_1 := \max \{ ||a||_{\infty} |2^{-1} \lambda^{-1} + 8 \lambda^2|, ||B||R + |H|R, R|H|a_0^{-1} \},
\]
\[
||B|| = \sup_{x \in \mathcal{M}} |B_x|, \text{ and } |B_x| = \sup_{v \in T_x \mathcal{M}, |v|=1} |B_x v|.
\]

5.2.8. **Intrinsic “cut-off”**. It remains to estimate the quantity \( \int_0^T \int_{\mathcal{M}_2} |\nabla T u|^2 \, d\mathcal{M} \, dt \) in terms of the damping term \( \int_0^T \int_{\mathcal{M}} [a(x)||g(u_t)||^2 + a(x)||u_t||^2] \, d\mathcal{M} \, dt \). For this purpose we have to built a “cut-off” function \( \tilde{\eta} \) on a specific neighborhood of \( \mathcal{M}_2 \). First of all, define \( \tilde{\eta} : \mathbb{R} \to \mathbb{R} \) such that
\[
\tilde{\eta}(x) = \begin{cases} 
1 & \text{if } x \leq 0 \\
(x-1)^2 & \text{if } x \in [1/2, 1] \\
0 & \text{if } x > 1
\end{cases}
\]
and it is defined on \((0, 1/2)\) in such a way that \( \tilde{\eta} \) is a non-increasing function of class \( C^1 \).

For \( \varepsilon > 0 \), set \( \tilde{\eta}_\varepsilon(x) := \tilde{\eta}(x/\varepsilon) \).

It is straightforward that there exists a constant \( M \) which does not depend on \( \varepsilon \) such that
\[
\frac{||\tilde{\eta}'_\varepsilon(x)||}{\tilde{\eta}_\varepsilon(x)} \leq \frac{M}{\varepsilon^2}
\]
for every \( x < \varepsilon \).

Now, let \( \varepsilon > 0 \) be such that
\[
\tilde{\omega}_\varepsilon := \{ x \in \mathcal{M}; d(x, \bigcup_{i=1}^k \partial \mathcal{M}_{0i}) < \varepsilon \}
\]
is a tubular neighborhood of \( \bigcup_{i=1}^k \partial \mathcal{M}_{0i} \) and \( \omega_\varepsilon := \tilde{\omega}_\varepsilon \cup \mathcal{M}_2 \) is contained in \( \mathcal{M}_* \). Define \( \eta_\varepsilon : \mathcal{M} \to \mathbb{R} \) as
\[
\eta_\varepsilon(x) = \begin{cases} 
1 & \text{if } x \in \mathcal{M}_2 \\
\tilde{\eta}_\varepsilon(d(x, \mathcal{M}_2)) & \text{if } x \in \omega_\varepsilon \setminus \mathcal{M}_2 \\
0 & \text{otherwise.}
\end{cases}
\]
It is straightforward that \( \eta_\varepsilon \) is a function of class \( C^1 \) on \( \mathcal{M} \) due to the smoothness of \( \partial \mathcal{M}_2 \) and \( \partial \omega_\varepsilon \). Notice also that
\[
(5.39) \quad \frac{||\nabla_T \eta_\varepsilon(x)||}{\eta_\varepsilon(x)} = \frac{||\tilde{\eta}_\varepsilon'(d(x, \mathcal{M}_2))||}{\tilde{\eta}_\varepsilon(d(x, \mathcal{M}_2))} \leq \frac{M}{\varepsilon^2}
\]
for every \( x \in \omega_\varepsilon \setminus \mathcal{M}_2 \). In particular, \( ||\nabla_T \eta_\varepsilon|| \in L^\infty(\omega_\varepsilon) \).
Taking $\xi = \eta_\varepsilon$ in the identity (5.33) we obtain

\begin{equation}
\int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |\nabla T u|^2 d\mathcal{M} dt \\
= - \left[ \int_{\omega_\varepsilon} u_t u \eta_\varepsilon d\mathcal{M} \right]^T_0 + \int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |u_t|^2 d\mathcal{M} \\
- \int_0^T \int_{\omega_\varepsilon} u(\nabla T u \cdot \nabla T \eta_\varepsilon) d\mathcal{M} dt - \int_0^T \int_{\omega_\varepsilon} a(x) g(u_t) u \eta_\varepsilon d\mathcal{M} dt.
\end{equation}

After some estimates we arrive to the following inequality

\begin{equation}
\frac{1}{2} \int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |\nabla T u|^2 d\mathcal{M} dt \\
\leq |\mathcal{Y}| + \frac{\lambda_1^{-1} ||a||_{L^\infty(\mathcal{M})}}{4\alpha} \int_0^T \int_{\mathcal{M}} a(x) |g(u_t)|^2 d\mathcal{M} \\
+ 2\alpha \int_0^T E(t) dt + \frac{M}{2\varepsilon^2} \int_0^T \int_{\omega_\varepsilon} |u_t|^2 d\mathcal{M} dt, \\
+ a_0^{-1} \int_0^T \int_{\mathcal{M}} a(x) u_t^2 d\mathcal{M} dt,
\end{equation}

where $\alpha > 0$ is an arbitrary number and

\begin{equation}
\mathcal{Y} := - \left[ \int_{\omega_\varepsilon} u_t u \eta_\varepsilon d\mathcal{M} \right]^T_0.
\end{equation}

Thus, combining (5.41) and (5.38), having in mind that

\begin{equation}
\frac{1}{2} \int_0^T \int_{\omega_\varepsilon} |\nabla T u|^2 d\mathcal{M} dt \leq \frac{1}{2} \int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |\nabla T u|^2 d\mathcal{M} dt
\end{equation}

and choosing $\alpha = 1/16C_1$ we deduce

\begin{equation}
\frac{1}{4} \int_0^T E(t) dt \leq |\chi| + 2C_1 |\mathcal{Y}|
\end{equation}

\begin{equation}
+ C_2 \int_0^T \int_{\mathcal{M}} [a(x) |g(u_t)|^2 + a(x) |u_t|^2] d\mathcal{M} dt \\
+ \frac{MC_1}{\varepsilon^2} \int_0^T \int_{\omega_\varepsilon} |u|^2 d\mathcal{M} dt,
\end{equation}

where $C_2 = \max \{C_1, 8C_1^2 \lambda_1^{-1} ||a||_{L^\infty(\mathcal{M})}, 2C_1 a_0^{-1} \}$.

On the other hand, the following estimate holds

\begin{equation}
|\chi| + 2C_2 |\mathcal{Y}| \leq C(E(0) + E(T))
\end{equation}

\begin{equation}
= C \left[ 2E(T) + \int_0^T \int_{\mathcal{M}} a(x) g(u_t) u_t d\mathcal{M} \right],
\end{equation}

where $C$ is a constant.
where $C$ is a positive constant which depends also on $R$. Then,

\begin{align}
T E(T) & \leq \int_0^T E(t) \, dt \\
& \leq C E(T) + C \left[ \int_0^T \int_\mathcal{M} [a(x) |g(u_t)|^2 + a(x)|u_t|^2] \, d\mathcal{M} \, dt \right] \\
& \quad + C \int_0^T \int_{\omega_x} |u_t|^2 \, d\mathcal{M} \, dt,
\end{align}  

where $C$ is a positive constant which depends on $a_0, ||a||_\infty, \lambda_1, R, |H|, ||B||$ and $\frac{M}{\varepsilon^2}$. Our aim is to estimate the last term on the RHS of (5.45). In order to do this let us consider the following lemma, where $T_0$ is a positive constant which is sufficiently large for our purpose.

**Lemma 5.3.** There exists a positive constant $C(E(0))$ such that if $u$ is the solution of (5) with weak initial data, we have

\begin{align}
\int_0^T \int_\mathcal{M} |u|^2 \, d\mathcal{M} \, dt & \leq C(E(0)) \left\{ \int_0^T \int_\mathcal{M} \left[ a(x) g^2(u_t) + a(x)u_t^2 \right] \, d\mathcal{M} \, dt \right\},
\end{align}  

for all $T > T_0$.

In order to prove the above lemma we argue by contradiction and it is essential to make use of the uniqueness result which comes from the Inverse Inequality or, more generally we can also employ Triggiani and Yao’s Uniqueness result (see [41]) in the proof. The details of the proof of lemma 5.3 can be found in Cavalcanti et. al. [9], [10] and we shall omit it here.

Inequalities (5.45) and (5.46) lead us to the following result.

**Proposition 5.2.2:** For $T > 0$ large enough, the solution $u$ of (5) satisfies

\begin{align}
E(T) & \leq C \int_0^T \int_\mathcal{M} \left[ a(x) |u_t|^2 + a(x) |g(u_t)|^2 \right] \, d\mathcal{M} \, dt
\end{align}  

where the constant $C = C(T, E(0), a_0, \lambda_1, R, ||B||, \frac{M}{\varepsilon^2})$.

From this point we are able to employ Lasiecka and Tataru’s method [20] in order to obtain the desired decay rates estimates (1.9).
5.2.9. Generalization of umbilical and conical surfaces - New regions. Invoking the second fundamental identity one more time now with $\xi = (m \cdot \nu)H$ we deduce

\[
\int_0^T \int_M (m \cdot \nu)H \left[ |u_t|^2 - |\nabla_T u|^2 \right] dMdt \\
= \left[ \int_M (m \cdot \nu)H u_t u dM \right]^T_0 \\
+ \int_0^T \int_M (\nabla_T u \cdot \nabla_T (m \cdot \nu)H) u dMdt \\
+ \int_0^T \int_M a(x) g(u_t) (m \cdot \nu)H u dMdt.
\]

(5.48)

Substituting (5.48) in (5.34) we infer

\[
\left[ \int_M u_t m_T \cdot \nabla_T u dM \right]^T_0 + \frac{1}{2} \left[ \int_M u_t u dM \right]^T_0 \\
+ \int_0^T E(t) dt + \int_0^T \int_M a(x) g(u_t) (m_T \cdot \nabla_T u) dMdt \\
+ \frac{1}{2} \int_0^T \int_M a(x) g(u_t) u dMdt = - \left[ \int_M (m \cdot \nu)H u_t u dM \right]^T_0 \\
- \int_0^T \int_M (\nabla_T u \cdot \nabla_T (m \cdot \nu)H) u dMdt \\
- \int_0^T \int_M a(x) g(u_t) (m \cdot \nu)H u dMdt. \\
- \int_0^T \int_M (m \cdot \nu)(\nabla_T u \cdot B \cdot \nabla_T u) dMdt.
\]

(5.49)

It is convenient to observe that the novelty above are the two terms $I_1 = - \int_0^T \int_M (\nabla_T u \cdot \nabla_T (m \cdot \nu)H) u dMdt$ and $I_2 = - \int_0^T \int_M (m \cdot \nu)(\nabla_T u \cdot B \cdot \nabla_T u) dMdt$. The analysis of the rest of the terms in (5.49) remains the same as we have considered before.

**Analysis of $I_1$**

\[
- \int_0^T \int_M (\nabla_T u \cdot \nabla_T (m \cdot \nu)H) u dMdt. \\
\text{We have, for an arbitrary } \varepsilon > 0, \\
|I_1| \leq \varepsilon \int_0^T E(t) dt + C_\varepsilon \int_0^T \int_M |u|^2 dMdt.
\]

The remaining term $C_\varepsilon \int_0^T \int_M |u|^2 dMdt$ can be absorbed by use of unique continuation property due to Triggiani and Yao [41] which is valid for general compact manifolds with boundary. It remains to analyse the term $I_2$, where the new and better geometric impositions will appear.

**Analysis of $I_2$**

\[
- \int_0^T \int_M (m \cdot \nu)(\nabla_T u \cdot B \cdot \nabla_T u) dMdt.
\]

Setting $\nabla_T u = (\xi, \eta)$ the coordinates of $\nabla_T u$ in the basis $\{e_1, e_2\}$, for each $x \in \mathcal{M}$, we deduce that

\[
\nabla_T u \cdot B \cdot \nabla_T u = k_1 \xi^2 + k_2 \eta^2.
\]
5.2.10. **New Geometric Conditions.** The sub-surface $M_0$ without damping must have non-negative Gaussian curvature, that is, $K = k_1 k_2 \geq 0$, with $k_1, k_2 \leq 0$ connected, and the closure of the Gauss map must be contained in an open semi-sphere (the last condition is required in order to guarantee that $x - x^0(x) \cdot \nu(x) \leq 0$ for all $x \in M_0$. The above geometric condition, now in terms of the **Gaussian curvature** $K = k_1 k_2$ instead of the **Mean curvature** $H = \frac{k_1 + k_2}{2}$ allow us to generalize our previous results. However, observe that we strongly need a **Unique Continuation Property based on Carleman estimates** which has been proved by Triggiani and Yao [41] for wave propagation on compact manifolds. Note that umbilical and conical sub-surfaces satisfy the above condition. In addition, we can consider new sub-surfaces without damping. See sub-subsections below.

![Figure 4](image1)

**Figure 4.**

5.2.11. **cylindrical surfaces** $K = 0$ (where $(x - x_0) \cdot \nu \leq 0$) can be also considered without damping.

5.2.12. **Torus** - we can avoid damping where $m(x) \cdot \nu(x) \leq 0$ and $K \geq 0$.

![Figure 5](image2)

**Figure 5.**

It is important to be mentioned that the techniques developed based on multipliers can be naturally extended for a finite number of observers $x_1, \cdots, x_n$ in connection with a finite number of *disjoint regions satisfying our geometrical impositions* $U_1, \cdots, U_n$ (see figures 6 and 8). Indeed, for the sake of simplicity let us consider the simple case where we have just two observers located at $x_1$ and $x_2$ and $U_1$ and $U_2$ are umbilical. Thus, it is sufficient to make use of the multiplier $q \cdot \nabla_T u$ where $q$ is defined by

$$q(x) := \begin{cases} x - x_i & \text{if } x \in U_i, \ i = 1, 2, \\ \text{smoothly extended in } M \setminus (U_1 \cup U_2) \end{cases}$$

(5.50)
accordingly the figure 6.

![Figure 6](image)

**Figure 6.**

Observe that if we consider $x_1$ and $x_2$ opposite with respect to the center of the sphere and sufficiently far from each other, the damping can be made effective in an arbitrarily small neighborhood of the meridian. This almost reaches the sharp result for the linear case due to Bardos, Lebeau and Rauch [2]. However, note that we have a nonlinear and localized damping. In addition, we can extend our results for the semi-linear wave equation as well having in mind we need a unique continuation property based on Carleman estimates.

![Figure 7](image)

**Figure 7.** One of the regions of geometric control condition (in red) intercepts all the geodesics on the sphere (in black).

5.2.13. Sharpness of the Geometric Control Condition.

5.2.14. If $A$ and $A'$ are antipodal points the damping can be reduced as $A$ and $A'$ go to infinity.

5.2.15. Sharpness of Microlocal Analysis when compared to Radial Multipliers. From the above considerations, there is no doubt regarding the sharpness of the microlocal analysis (for the wave propagation) when compared with the multiplier technique (compare figures 6, 7, 8 and 9), even for the case one has a nonlinear distributed damping:

$$u_{tt} - \Delta u + a(x)g(u_t) = 0, \text{ in } M \times (0, \infty).$$

On the other hand, the multiplier method is still a powerful tool for treating nonlinear equations. To end this we would like to comment briefly in the next section the results contained in Cavalcanti et. al. [9]. In this work the authors developed an intrinsic multiplier
as follows. The main goal of this intrinsic multiplier is to improve considerably their previous result reducing arbitrarily the volume of the region where the dissipative effect lies.

5.3. Intrinsic Multipliers. Denoting by $g$ the Riemannian metric induced on $\mathcal{M}$ by $\mathbb{R}^3$, they prove that for each $\epsilon > 0$, there exist an open subset $V \subset \mathcal{M}$ and a smooth function $f : \mathcal{M} \to \mathbb{R}$ such that

$$\text{meas}(V) \geq \text{meas}(\mathcal{M}) - \epsilon, \quad \text{Hess} f \approx g$$

on $V$ and $\inf_{x \in V} |\nabla f(x)| > 0$. This new intrinsic multiplier $\nabla f(x)$, instead of the previous one $m(x) = x - x^0$, will play a crucial role when establishing the sharpness of the desired uniform decay rates of the energy.

The first step is to consider an identity we have already presented, namely.

**Proposition 5.4.** Let $\mathcal{M} \subset \mathbb{R}^3$ be an oriented regular compact surface without boundary and $q$ a vector field of class $C^1$. Then, for every regular solution $u$ of (5) we have the
following identity:

\[
\left[ \int_{\mathcal{M}} u_t q \cdot \nabla_T u d\mathcal{M} \right]_0^T \\
+ \frac{1}{2} \int_0^T \int_{\mathcal{M}} (\text{div} q) \left\{ |u_t|^2 - |\nabla_T u|^2 \right\} d\mathcal{M} dt \\
+ \int_0^T \int_{\mathcal{M}} \nabla_T u \cdot \nabla_T q \cdot \nabla_T u d\mathcal{M} dt \\
+ \int_0^T \int_{\mathcal{M}} a(x)(u_t)(q \cdot \nabla_T u) d\mathcal{M} dt = 0.
\]

The proof is based on multiplying the equation by \( q \cdot \nabla_T u \) and integration by parts. Employing the above identity with \( q(x) = \nabla_T f \) where \( f : \mathcal{M} \to \mathbb{R} \) is a \( C^3 \) function to be determined later, we infer

\[
\left[ \int_{\mathcal{M}} u_t \nabla_T f \cdot \nabla_T u d\mathcal{M} \right]_0^T \\
+ \frac{1}{2} \int_0^T \int_{\mathcal{M}} \Delta_{\mathcal{M}} f \left\{ |u_t|^2 - |\nabla_T u|^2 \right\} d\mathcal{M} dt \\
+ \int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot \text{Hess}(f) \cdot \nabla_T u) d\mathcal{M} dt \\
+ \int_0^T \int_{\mathcal{M}} a(x)(u_t)(\nabla_T f \cdot \nabla_T u) d\mathcal{M} dt = 0.
\]

**Lemma 5.5.** Let \( u \) be a weak solution to problem (5) and \( \xi \in C^1(\mathcal{M}) \). Then

\[
\left[ \int_{\mathcal{M}} u_t \xi u d\mathcal{M} \right]_0^T = \int_0^T \int_{\mathcal{M}} \xi |u_t|^2 d\mathcal{M} dt \\
- \int_0^T \int_{\mathcal{M}} \xi |\nabla_T u|^2 d\mathcal{M} dt \\
- \int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot \nabla_T \xi) u d\mathcal{M} dt \\
- \int_0^T \int_{\mathcal{M}} a(x)(u_t) \xi u d\mathcal{M} dt.
\]

The proof is based in multiplying the equation by \( \xi u \) and integrating by parts.
Substituting $\xi = \alpha > 0$ in the last identity and combining the obtained result with the previous identity we deduce

$$\int_0^T \int_M \left( \frac{\Delta_M f}{2} - \alpha \right) |u_t|^2 \, dM \, dt.$$

$$+ \int_0^T \int_M \left[ (\nabla_T u \cdot \text{Hess}(f) \cdot \nabla_T u) + \left( \alpha - \frac{\Delta_M f}{2} \right) |\nabla_T u|^2 \right] \, dM \, dt
$$

$$= - \left[ \int_M u_t \nabla_T f \cdot \nabla_T u \, dM \right]_0^T - \alpha \left[ \int_M u_t \, u \, dM \right]_0^T
$$

$$- \alpha \int_0^T \int_M a(x) \, g(u_t) \, u \, dM \, dt
$$

$$- \int_0^T \int_M a(x) \, g(u_t)(\nabla_T f \cdot \nabla_T u) \, dM \, dt.$$

This is the precise moment where the properties of function $f$ play an important role. Note that what we just need is to find a subset $V$ of $\mathcal{M}$ such that

$$C \int_0^T \int_V \left[ u_t^2 + |\nabla_T u|^2 \right] \, dM \, dt
$$

$$\leq \int_0^T \int_V \left( \frac{\Delta_M f}{2} - \alpha \right) |u_t|^2 \, dM \, dt
$$

$$+ \int_0^T \int_V \left[ (\nabla_T u \cdot \text{Hess}(f) \cdot \nabla_T u) + \left( \alpha - \frac{\Delta_M f}{2} \right) |\nabla_T u|^2 \right] \, dM \, dt,$$

for some positive constant $C$, provided that $\alpha$ is suitably chosen.

Assuming, for a moment, that the last inequality holds we obtain

$$(5.51) \quad 2C \int_0^T E(t) \, dt \leq C \int_0^T \int_{\mathcal{M}\setminus V} \left[ u_t^2 + |\nabla_T u|^2 \right] \, dM \, dt
$$

$$+ \left[ \int_M u_t \nabla_T f \cdot \nabla_T u \, dM \right]_0^T + \alpha \left[ \int_M u_t \, u \, dM \right]_0^T
$$

$$+ \left| \int_0^T \int_M a(x) \, g(u_t) \, u \, dM \, dt \right|
$$

$$+ \left| \int_0^T \int_M a(x) \, g(u_t)(\nabla_T f \cdot \nabla_T u) \, dM \, dt \right|.$$

The above inequality is controlled considering a standard procedure. The main idea behind this procedure is to consider the dissipative area, namely, $\mathcal{M}_*$, containing the set $\mathcal{M}\setminus V$. It is important to observe that $\mathcal{M}_*$ is as small as big $V$ can be. From (5.51) we conclude the desired decay rate estimates as considered previously. So, it is enough to prove the main inequality (5.51) for $V$ as large as possible. The complete proof can be found in Cavalcanti et. al. [9], [10].
The next steps are devoted to the construction of a function \( f \) as well as a subset \( V \) of \( \mathcal{M} \) such that the desired inequality holds.

### 5.3.1. Construction of function \( f \) locally

Let \( \mathcal{M} \) be a compact \( n \)-dimensional Riemannian manifold (without boundary) with Riemannian metric \( g \) of class \( C^2 \). Let \( \nabla \) denote the Levi-Civita connection. Fix \( p \in \mathcal{M} \). Our aim is to construct a function \( f : V_p \to \mathbb{R} \) such that the desired inequality holds.

We begin with an orthonormal basis \((e_1, \ldots, e_n)\) of \( T_p \mathcal{M} \). Define a normal coordinate system \((x_1, \ldots, x_n)\) in a neighborhood \( \tilde{V}_p \) of \( p \) such that \( \frac{\partial}{\partial x_i}(p) = e_i(p) \) for every \( i = 1, \ldots, n \). It is well known that in this coordinate system we have that \( \Gamma^k_{ij}(p) = 0 \), where \( \Gamma^k_{ij} \) are the Christoffel symbols with respect to \((x_1, \ldots, x_n)\).

The Hessian with respect to \((x_1, \ldots, x_n)\) is given by

\[
\text{Hess}_f \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial f}{\partial x_k}.
\]

The Laplacian of \( f \) is the trace of the Hessian with respect to the metric \( g \). If \( g_{ij} \) denote the components of the Riemannian metric with respect to \((x_1, \ldots, x_n)\) and \( g^{ij} \) are the components of the inverse matrix of \( g_{ij} \), then the Laplacian of \( f \) is given by

\[
\Delta f = \sum_{i,j} g^{ij} \text{Hess}_f \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).
\]

Consider the function \( f : \tilde{V}_p \to \mathbb{R} \) defined by

\[
f(x) = x_1 + \frac{1}{2} \sum_{i=1}^{n} x_i^2.
\]

It is immediate that \( \Delta f(p) = n \) and \( |\nabla f| = 1 \). Moreover, \( \text{Hess}_f(p) = g(p) \), which implies that \( \text{Hess}_f(p)(v, v) = |v|^2_p \).

We are interested in finding a neighborhood \( V_p \subset \tilde{V}_p \) of \( p \) and a strictly positive constant \( C \) such that

\[
C \int_0^T \int_{V_p} (|\nabla u|^2 + u_t^2) \, d\mathcal{M} dt
\]

\[
\leq \int_0^T \int_{V_p} \left[ \text{Hess}_f(\nabla u, \nabla u) + \left( \alpha - \frac{\Delta f}{2} \right) |\nabla u|^2 + \left( \frac{\Delta f}{2} - \alpha \right) u_t^2 \right] \, d\mathcal{M} dt,
\]

for some \( \alpha \in \mathbb{R} \).

We claim that if we consider \( \alpha = \frac{n}{2} - \frac{1}{2} \) and \( C = 1/4 \) we obtain the desired inequality, what means that it is enough to prove that there exist \( V_p \subset \tilde{V}_p \) verifying

\[
\int_0^T \int_{V_p} \text{Hess}_f(\nabla u, \nabla u) + \left( \frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2} \right) |\nabla u|^2 d\mathcal{M} dt \geq 0
\]
\[ \int_0^T \int_{\tilde{V}_p} \left( \frac{\Delta f}{2} - \frac{n}{2} + \frac{1}{4} \right) u^2 dM dt \geq 0. \]

In order to prove the existence of a subset \( V_p \subset \tilde{V}_p \) where the first inequality holds, let \( \theta_1 \) be the smooth field of symmetric bilinear form on \( \tilde{V}_p \) defined as

\[ \theta_1(X,Y) = Hess f(X,Y) + \left( \frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2} \right) g(X,Y) \]

where \( X \) and \( Y \) are vector fields on \( \tilde{V}_p \).

It is clearly a positive definite bilinear form on \( \tilde{V}_p \) since

\[ Hess f(p)(X,Y) = g(p)(X,Y) \]

and

\[ \theta_1(p)(X,Y) = \frac{1}{4} g(p)(X,Y). \]

Therefore, there exist a neighborhood \( \tilde{V}_p \) such that \( \theta_1 \) is positive definite and

\[ \int_0^T \int_{\tilde{V}_p} Hess f(\nabla u, \nabla u) + \left( \frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2} \right) |\nabla u|^2 dM dt \geq 0. \]

To prove the existence of \( \tilde{V}_p \subset \tilde{V}_p \) such that the desired inequality holds is easier. It is enough to notice that at \( p \) we have that

\[ \left( \frac{\Delta f(p)}{2} - \frac{n}{2} + \frac{1}{4} \right) = \frac{1}{4} \]

and the existence of \( \tilde{V}_p \subset \tilde{V}_p \) is immediate. Furthermore we can eventually choose a smaller \( V_p \) such that \( \inf_{x \in V_p} |\nabla f(x)| > 0 \). Therefore the existence of \( V_p \subset \tilde{V}_p \) such that \( \inf_{x \in V_p} |\nabla f(x)| > 0 \) and (5.52) holds is settled.

In what follows, \( \tilde{V} \) denotes the closure of \( V \) and \( \partial V \) denotes the boundary of \( V \). When \( \tilde{V} \subset W \) is bounded, we say that \( V \) is compactly contained in \( W \) and we denote by \( V \subset \subset W \).

**Theorem 5.6.** Let \((\mathcal{M}, g)\) be a two dimensional Riemannian manifold. Then, for every \( \epsilon > 0 \), there exist a finite family \( \{V_i\}_{i=1,\ldots,k} \) of open sets with smooth boundary, smooth functions \( f_i: \tilde{V}_i \to \mathbb{R} \) and a constant \( C > 0 \) such that

1. The subsets \( \tilde{V}_i \) are pairwise disjoint;
2. \( \vol(\bigcup_{i=1}^k V_i) \geq \vol(M) - \epsilon; \)
3. Inequality (5.52) holds for every \( f_i; \)
4. \( \inf_{x \in \tilde{V}_i} |\nabla f(x)| > 0 \) for every \( i = 1, \ldots, k. \)

In order to prove this Theorem let us consider the following steps:

First of all, it is possible to get open subsets \( \{\tilde{W}_j\}_{j=1,\ldots,s} \) with smooth boundaries and a family of smooth functions \( \{f_j: \tilde{W}_j \to \mathbb{R}\}_{j=1,\ldots,s} \) such that \( \{\tilde{W}_j\}_{j=1,\ldots,s} \) is a cover of \( \mathcal{M} \) and each \( f_j \) satisfies Inequality (5.52). Moreover, we can choose \( \tilde{W}_j \) in such a way that their boundaries intercept themselves transversally and three or more boundaries do not intercept themselves at the same point.
Set by $A := \bigcup_{j=1}^{k} \partial \tilde{W}_j$. Then, $\mathcal{M}\setminus A$ is a disjoint union of connected open sets $\bigcup_{i=1}^{k} W_i$ such that $\partial W_i$ is a piecewise smooth curve.

Each $W_i$ is contained in some $\tilde{W}_j$. Therefore, for each $W_i$, choose a function $\hat{f}_i := \tilde{f}_j|_{W_i}$.

The open subsets $V_i$, $i = 1, \ldots, k$, we are looking for are subsets of $W_i$. We can choose them in such a way that

1. $V_i \subset W_i$;
2. $\partial V_i$ is smooth;
3. $\text{vol}(W_i) - \text{vol}(V_i) < \epsilon/k$.

Finally, if we set $f_i = \hat{f}_i|_{V_i}$, we prove the theorem.

**Theorem 5.7.** Let $(\mathcal{M}, g)$ be a two-dimensional Riemannian manifold. Fix $\epsilon > 0$. Then, there exist a smooth function $f : M \to \mathbb{R}$ such that inequality (5.52) and the condition $\inf_{x \in V_i} |\nabla f(x)| > 0$ hold in a subset $V$ with $\text{vol}(V) \geq \text{vol}(\mathcal{M}) - \epsilon$.

In order to give an idea of the proof, consider Theorem 5.6 and the constructions made in its proof. Denote $\lambda := \min_{i \neq j} \text{dist}(V_i, V_j) > 0$. Consider a tubular neighborhood $V_\delta$ of $V = \bigcup_{i=1}^{k} V_i$ of the points whose distance is less than or equal to $\delta < \lambda/4$. Then, it is possible to define a smooth (cut-off) function given by

$$\eta : \mathcal{M} \to \mathbb{R}$$

such that

$$\eta(x) = \begin{cases} 1 & \text{if } x \in V \\ 0 & \text{if } x \in \mathcal{M}\setminus V_\delta \\ \text{between 0 and 1 otherwise} \end{cases}$$

Now, notice that $f : \mathcal{M} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \hat{f}_i(x)\eta(x) & \text{if } x \in W_i \\ 0 & \text{otherwise} \end{cases}$$

is smooth and satisfy inequality (5.52) and the condition $\inf_{x \in V} |\nabla f(x)| > 0$. In addition, the inequality $\text{vol}(V) \geq \text{vol}(\mathcal{M}) - \epsilon$ holds, which settles the theorem.

$$\text{meas}(V) \geq \text{meas}(\mathcal{M}) - \epsilon, \ \text{Hess} f \approx g$$

on $V$ and $\inf_{x \in V} |\nabla f(x)| > 0$. This new intrinsic multiplier $\nabla f(x)$, instead of the previous one $m(x) = x - x^0$, play a crucial role when establishing the desired uniform decay rates of the energy.

Although the *intrinsic result* is sharp with respect to the volume where the damping acts, we do not have any control about the regions that can be left free of damping. The connected disjoint components of $V$ can be extremely small. See figure below.

In the other direction, the *external vision* states that some umbilical domains of surfaces in $\mathbb{R}^3$ can be left free of damping.

Therefore the next step is to combine the ideas of both techniques and try to put the damping in a arbitrarily small domain, but in such a way that domains with interesting properties can be left free of damping. Combining and the techniques developed in [9] and
$M_0$ is a non-dissipative area (in white) arbitrarily large while the demarcated area (in black) contains dissipative effects and can be considered arbitrarily small, both totally distributed on $M$.

Figure 10

[10] we can reduce arbitrarily the superficial measure of the dissipative area. Here the vector field $q$ is defined as

$$q(x) := \begin{cases} x - x_i & \text{if } x \in U_i, i = 1, 2, \\ \nabla f(x), Hess(f) \approx g, & \text{if } x \text{ is in some small white domain of } D \\ \text{smoothly extended otherwise} \end{cases}$$

where $g$ is the Riemannian metric on $\mathcal{M}$ (see Figure 11).

Figure 11
Finally, we would like to emphasize that in Cavalcanti et. al [9] the authors have generalized (in some sense) and just for the wave equation, the results presented previously for $n$-dimensional compact Riemannian manifolds $(M, g)$ with or without boundary. They proceed as follows:

1. The authors prove that for every $x \in M$ (including the case $x \in \partial M$), there exists a neighborhood that can be left without damping;
2. They prove that a very precise portion of radially symmetric domains can be left without damping (see figure 12);
3. Let $\varepsilon > 0$ and $V_1, \ldots, V_k$ be domains as in (i) and (ii) which closures are pairwise disjoint. We prove that there exist a $V \supset \bigcup_{i=1}^k V_i$ that can be left without damping and such that $\text{meas}(V) \geq \text{meas}(M) - \varepsilon$ and $\text{meas}(V \cap \partial M) \geq \text{meas}(\partial M) - \varepsilon$.

A radially symmetric region $\Omega$

Figure 12. The demarcated region $M \setminus V$ (in black) illustrates the damped region on the compact manifold $M$ with boundary $\partial M$, which can be considered as small as desired. $\Omega$ is radially symmetric region without damping. The measure of $\partial M \cap (M \setminus V)$ can also be arbitrarily small

5.4. Final Conclusion. The hypothesis of the damping region in terms of the rays of the geometric optics has close relationship when it is compared with the hypothesis of the existence of a “nice” vector field $q$. In one hand, the results in terms of the ray of the geometric optics or geometric control condition (GCC) are more general for the linear wave equation with a nonlinear dissipative term locally distributed:

$$\partial_t^2 u - \Delta u + a(x)g(\partial_t u) = 0, \text{ in } M \times (0, \infty).$$

However the results exploiting the multipliers also consider the semi-linear wave equation subject to a nonlinear and localized dissipative mechanism and give explicitly examples of regions that can be left without damping, which can be a difficult task if we use the hypothesis on the ray of geometric optics on a general compact Riemannian manifold. In our opinion, there are plenty of space left for further studies about the relationships between these two different kind of hypothesis.

On the other hand, for other equations as plates, Schrödinger equations, etc ... the Geometric Control Condition does not give us sharp results in terms of reducing arbitrarily the region where the damping acts. For these equations, even if we consider a nonlinear
and localized dissipation on the Riemannian manifold, to assume that the observability inequality holds for the linear equation, it seems to be the best choice still.

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