Effective dynamics for chaos synchronization in networks with time-varying topology

Romeu M. Szmoski, Rodrigo F. Pereira, Sandro E. de Souza Pinto

Departamento de Física, Universidade Estadual de Ponta Grossa, 84030-900 Ponta Grossa, PR, Brazil

Abstract

A coupled map lattice whose topology changes at each time step is studied. We show that the transversal dynamics of the synchronization manifold can be analyzed by the introduction of effective dynamical quantities. These quantities are defined as weighted averages over all possible topologies. We demonstrate that an ensemble of short time observations can be used to predict the long-term behavior of the lattice. Finally, we point out that it is possible to obtain a lattice with constant topology in which the dynamical behavior is asymptotically identical to one of the time-varying topology.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

In nature, we see objects that interact in various ways. These objects define sets, called networks, that present a specific structure according to each situation observed. When objects are identical an important phenomenon can occur: the complete synchronization of chaos [1]. This kind of synchronization is a phenomenon in which all elements of a network present the same states in the same time. Neurons in the brain [2], groups of individuals [3] and the world wide web [4] are examples of systems that can exhibit synchronization. These three systems, among others, have a complex topology as part of its structure, and, for each specific topology different properties are observed [5]. Random, small-world and scale-free are the most prominent among the networks that appear in literature [3,6–8]. In real systems, such as topologies can coexist in the same instant, or change as time passes [9–11].

Synchronization can occur in the systems cited above, as well as in models using topologies above mentioned. A recent work [12] has studied the synchronization of clusters in genetic oscillators with external disturbances, in which the time delay is taken into account, in order to simulate the delay in the transmission of information in real systems. Aiming to optimize synchronization between coupled chaotic systems the authors of Ref. [13] present an adaptive strategy that has two parallel objectives: to minimize the intensity of coupling optimization and convergence speed. These two studies show more realistic models and aim to capture the complexity of real systems. In a paper in which it uses the topology of the cat brain authors study the role of synchronization, among other attributes of a complex network, to characterize the most central nodes of a network [14].

Using the synchronization, we present a method in order to predict the long-term behavior of networks whose topology varies over time and present long-range interactions [15]. The main idea of this work is the statement of effective quantities static quantities that replace those that vary over time for the analysis of the transversal dynamics to the synchronization manifold. These quantities, obtained by calculating a weighted average over the set of possible states, can be used to verify...
the ability of synchronization of a network as well as for derive the quantities that are defined on the subspace of synchronization, for example, the conditional Lyapunov exponents and upper bond of mutual information [16]. As a parameter for checking the ability of the network to synchronize, the effective quantities adds to the pre-existing approaches as the Hajnal diameter [17], and the stability method in a connection graph [18] among others. Another advantage of our approach is to allow the construction of a lattice with a static coupling which, for synchronization purposes, exhibits behavior consistent with the networks with time-varying topology. We say that such effective coupling represents the effective dynamics of the system. Therefore, the main contribution of this paper is to determine the circumstances under which a network with variable topology, reaches the synchronization, and from there build a network with effective coupling, which can be determined from the effective quantities, which regains its dynamics.

We hypothesized that each realization of the original time-varying network, for a long enough observation period, can be replaced by an effective system. Moreover, we expect that any network with time-varying topology can be represented by a circulant effective matrix, provided that the network is periodic and that the probability of interaction between elements is invariant over translations. Knowing the Lyapunov exponents that characterize the transversal dynamics to the synchronized state, we can find a good estimate for the effective coupling between the elements of the system under study.

The article has six main sections including the introduction. Firstly, we shall present the class of coupled map lattice, with time-varying topology, that is the subject of this work. The next section outlines the mathematical tools for calculating the effective quantities. This will be followed by the derivation of the expression for such quantities and the construction of the effective system. The next section will present a application of our results in a coupled map lattice whose coupling between the elements is semirandomly determined: the coupling probability between two elements in a given instant depends algebraically on their distance in the lattice. This coupling form was studied in Refs. [19,20] as a model of a network of neurons. Our conclusions are left to the last section.

2. Coupled map lattice

We examined a network called coupled map lattice (CML) [21], which is defined in the following way: let \( g : \omega \to \omega \) be an axiom-A\(^1\) \( d\)-dimensional map defined in \( \omega \subset \mathbb{R}^d \). Let \( \mathbf{F} : \Omega \to \Omega \) be a lattice with \( N \) coupled maps in the form \( g \), with \( \Omega = \omega^N \). This lattice reads

\[
\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n, n) = \mathbf{G}_n(\mathbf{y}_n),
\]

in which \( \mathbf{y}_n = \begin{bmatrix} x_0^n & \ldots & x_N^n \end{bmatrix} \), \( x_m^n \) defines the state of the \( m \)th site, \( \mathbf{f}(m)(\mathbf{y}_n) = g(x_m^n) \), and \( \mathbf{G}_n \) is a \( dN \times dN \) matrix that depends on discrete time \( n \), in an explicit way. This is the coupling matrix.

2.1. Topology

We shall restrict the lattice to the linear couplings, as we can see in the Eq. (1), and consider the lattice topology varying stochastically with time. However, the following conditions must be observed:

(i) the lattice is a periodic one;
(ii) the topology at each time interval is invariant under translations in the lattice.

Since there are constraints about the topology, one can call this form of semirandom coupling. The above conditions imply \( \mathbf{G}_n \) is a circulant matrix. Due to periodicity imposed to the topology, there is no reason to define a preferential direction (say, to the left or to the right) and we also require the topology to be a symmetric one. These features do not depend on the probability rule that defines each possible coupling matrix. The condition that the phase space \( \Omega \) is a direct product of \( N \) subspaces \( \omega \) is satisfied if, and only if,

\[
[\mathbf{G}_n]_{qw} \geq 0, \quad \sum_{m=0}^{dN-1} [\mathbf{G}_n]_{qwm} = 1, \quad (\forall q, w).
\]

Herein, \( q, w = 0, 1, \ldots, dN - 1 \). Thus, \( \mathbf{G}_n \) is also a stochastic matrix. We imposed also that the system (1) represents the coupling (direct or indirect) between all the \( N \) maps. That is, the state of an arbitrary site eventually influences all other elements in the lattice. If not, the system (1) can be splitted in two uncoupled subsystems. In this case, our results are applied separately to each subsystem. Mathematically, this means that the condition \( 0 < [\mathbf{G}_n]_{qw} < 1 \) must be satisfied, for at least one \( \mathbf{G}_n \), for each \( w \), in a such way that the infinite product of matrices \( \mathbf{G}_n \) typically results in a strictly positive matrix [23,24]. For instance, if each site is always coupled with its first neighbors, this condition is always satisfied. Here, we consider that the network topology uniquely defines the coupling matrix \( \mathbf{G}_n \), i.e. \( \mathbf{G}_n = \mathbf{G}_n(\mathbf{T}_n) \), being \( \mathbf{T}_n = \mathbf{1} + \mathbf{A}_n \) the topology matrix, \( \mathbf{1} \) the identity matrix and \( \mathbf{A} \) the usual adjacency matrix [25].

\(^1\) A map that satisfies the axiom-A is hyperbolic and mixing [22].
If we suppose that the rules that specify the coupling probabilities are well-defined and do not change over time, we have

\[
[T_n]_{0r} = [T_n]_{0(r+dN-r)} = \begin{cases} 
1 & \text{if } \xi \leq p_r, \\
0 & \text{if } \xi > p_r,
\end{cases} 
\]

\[
[T_n]_{qr} = [T_n]_{0([q mod dN])},
\]

in which \( r = 0, 1, \ldots, N' \), with \( N' \equiv \frac{dN-1}{2} \), \( \xi \) is random variable, and \( p_r \) is a parameter that determines the connectivity between sites 0 and \( r \). We remark that the connectivity distribution of each site is uniquely defined by the parameter \( p_r \). If \( \xi \) is \( \delta \)-correlated and with uniform probability distribution in [0, 1], then \( p_r \) defines the coupling probability. Therefore, the probability of occurrence of a specific coupling matrix depends only on entries of the topology matrix:

\[
\pi_k = \prod_{r=0}^{N'} (p_r[T_n]_{0r} + (1-p_r)(1-[T_n]_{0r})).
\]

If the conditions (i) and (ii) are fulfilled, there are, at most, \( K_N = 2^{\frac{dN}{2}+1} \) distinct matrices. For each \( p_r = 0 \) or \( p_r = 1 \), \( K_N \) is reduced by a factor 2. The evolution of the network is then obtained by the selection of \( G_n \) from the set \( \{G_n\}_{k=1}^{K_N} \), according the probability \( \pi_k \).

The translational invariance of the lattice topology implies all sites share the same connectivity degree \( K_n \) at the same instant \( n \). Along the temporal evolution of the lattice, \( K_n \) follows a Poissonian distribution, because we assume that the \( p_r \)'s are independent of each other. Different distributions can be obtained through a rescaling of the probabilities \( \pi_k \) following their respective connectivities.

### 3. Chaos synchronization and linear stability analysis

We are interested in studying the CML (1) from the point of view of the synchronization manifold stability. Such manifold is defined by

\[
S = \{ \mathbf{y} \in \Omega : \mathbf{s} \in \omega : \mathbf{x}^{(m)} \equiv \mathbf{s} \mod m \},
\]

that is, we are considering the complete amplitude synchronization - all sites share the same state at same time. We say that a trajectory \( \{\mathbf{y}_t\}_{t=0}^n \) achieves \( S \) at time \( n \) when, for any \( \delta > 0 \), we observe \( \text{dist}(\mathbf{y}_n, S) < \delta \). The mapping \( F \) keeps \( S \) invariant, and the dynamics in this subspace is determined by \( g \).

The stability of the synchronization manifold can be determined through the Lyapunov exponents’ spectrum of \( S \), calculated for a typical trajectory in this manifold. The \( m \)th Lyapunov exponent is defined by

\[
\lambda_m(s_0) = \lim_{n \to \infty} \frac{1}{n} \ln \|DF^n(s_0)e_m\|,
\]

in which \( D \equiv \frac{\partial}{\partial x} \) is the \( dN \)-dimensional Jacobian matrix, \( e_m \) is a typical unitary vector in \( \mathbb{R}^m \setminus \mathbb{R}^{m+1} \), and \( \mathbb{E}^0 = \Omega \). Since the dynamics in \( S \) is ruled by the map \( g \), which is, by assumption, mixing, \( \lambda_m(s_0) \) do not depends on \( s_0 \), apart from a set of null Lebesgue measure, and we have \( \lambda_m(s_0) \equiv \lambda_m \). The calculations yielded by \( (4) \) give \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{dN-1} \). The analysis of the stability of \( S \) is done by the examination of the greatest transversal Lyapunov exponent, denoted by \( \lambda_1 \), which is calculated in \( \Omega \setminus S \). In the case of networks whose topology varies over time, the scaling behavior of the Hajnal diameter of the product of coupling matrices enables us to obtain \( \lambda_1 \) [26,17].

Now, let us develop Eq. (4) for typical trajectories embedded in \( S \). For the sake of simplicity, we shall consider \( d = 1 \). Let \( J_n(y_0) \) be the Jacobian matrix applied in (1), which is calculated over \( \{\mathbf{y}_t\}_{t=0}^{n-1} \):

\[
J_n(y_0) = \prod_{t=0}^{n-1} G_t D G_t(y_t).
\]

In \( S \) we have \( D G_t(y_t) = \left( G_t H G_t \right)_{0N} \), where \( 0N \) is the identity \( N \times N \) matrix. Thus, the calculation of the Lyapunov exponents given by Eq. (4) yields

\[
\lambda_m = \lim_{n \to \infty} \frac{1}{n} \ln \left\| e^{D H n} \prod_{t=0}^{n-1} G_t e_m \right\|, \tag{5}
\]

in which \( \lambda_0 > 0 \) is the Lyapunov exponent of the map \( g(s) \).

Once \( G_t \) is a matrix satisfying the conditions (i), (ii) and Eq. (2), we can show that ²

² Our results remain valid under less restrictive conditions with respect to the possible coupling matrices. Theorems in Refs. [24,23] guarantee that the infinite product of matrices that are indecomposable and aperiodic converges to a matrix whose all rows are identical (matrix \( H \)). A matrix is indecomposable if it contains no submatrix of order \( N \) composed of zeros, whatever \( s > 0 \). If there is a finite sequence whose product results in a strictly positive matrix, i.e. all matrix elements are positive, the infinite product of indecomposable matrices converges to the matrix \( H \). This result was used by Lu et al. in Refs. [17,26] in the study of coupled map lattices in which the coupling is time-varying. They demonstrated the need for the union of all sets contains a spanning tree, that is, any pair of elements must be eventually coupled. We extend the applicability of expression (7) for all Lyapunov exponents of the synchronization subspace, from which we could build an effective system (autonomous and deterministic). The restrictions that we imposed on the topology of the network allowed us to derive analytically an expression for the effective quantities, Eq. (8).
\[ \lim_{n \to \infty} \prod_{k=0}^{n-1} G_k = H, \]

with \( |H|_{qw} = |H|_{0w} \), for all \( q \) and \( w \). The values labeled by \( |H|_{0w} \) depends on the observed coupling matrix chronological sequence, though it does not hold for the \( H \) form. \( H \) is a stochastic matrix, whose greatest eigenvalue is \( \eta_0 \equiv 1 \), which is associated to the eigenvector \( h_0 = \frac{1}{\sqrt{N}} [1 \cdots 1]^T \). All the others eigenvalues, associated with the eigenvectors \( h_q \ (q > 0) \) are equal to zero, \( \eta_q \equiv 0 \).

4. Results and discussion

4.1. The effective quantities

In the limit \( n \to \infty \), the Lyapunov vectors of \( S \), which are parallel to the eigendirections of the Jacobian matrix, are determined by \( h_n \). For this reason, the Lyapunov exponents are given by

\[ \lambda_m = \lim_{n \to \infty} \frac{1}{n} \ln \| e^{i\eta_n \hat{H}_m} \|, \]

with \( e_m = h_m \) in (5) (the index of \( h_m \) are chosen to provide a non-increasing spectrum).

For \( n \to \infty \) we have \( e^{i\eta_n \hat{H}_m} \to \infty \) and \( \eta_n \to 0 \ (m > 0) \). If we write

\[ \hat{\theta}_m = (\eta_m)^{-1/n}, \]  
\[ \lambda_m = \lim_{n \to \infty} \frac{1}{n} \ln \left\| \left( e^{i\eta_n \hat{\theta}_m} \right)^n \right\| = \lambda_U + \ln |\hat{\theta}_m|. \]  
\[ (7) \]

The quantities \( \lambda_m \) and \( (e^{i\theta_m}) \) represent the \( m \)th effective exponent and Lyapunov number, respectively, and we call \( \hat{\theta}_m \) the \( m \)th effective coupling eigenvalue. For \( n \gg 1 \) we have \( n\hat{\theta}_m \), which describe the time-\( n \) average behavior of the system. If \( \eta_m \) approaches zero faster than the exponential, the limit \( n \to \infty \) in Eq. (6) results \( \theta_m = 0 \), therefore, Eq. (7) gives \( \lambda_m = -\infty \).

Expression (6) gives a general and strong meaning to the effective coupling eigenvalues. The \( \hat{\theta}_m \ (m > 0) \) quantity is obtained by the decay rate to zero of the eigenvalues of the matrix \( H_m = \prod_{k=0}^{n-1} G_k \). Only one condition was imposed, namely \( \lim_{n \to \infty} H_n = H \). In other words, for any system in the form (1), in which all the possible coupling matrices satisfies (i), (ii) and Eq. (2), the results (6) and (7) are formally valid. Thus, in general, our results suggest that the effective dynamics of a network whose topology is time-varying can be obtained by the analytical or numerical calculation of \( \hat{\theta}_m \).

Since the conditions (i) and (ii) imply all coupling matrices are circulant, we can write down the exact expression for (6). This is possible because all circulant matrices share the same basis and, consequently, commute with each other. For this reason,

\[ \hat{\theta}_m = \prod_{k=1}^{K_\mu} \left( \Gamma^{(k)}_m \right)^{\pi_k}, \]

in which the productory extends over all possible coupling matrices, and \( \Gamma^{(k)}_m \) is the \( m \)th eigenvalue of the \( k \)th coupling matrix. Here and in what follows, the \( \Gamma^{(k)}_m \) (and the \( \hat{\theta}_m \)) are not indexed by their magnitudes, but according their respective eigenvectors whose components are \( h_q^{(k)} \propto \exp (-(2\pi i)qw/N) [27] \). Notice that, due symmetry, \( \Gamma^{(k)}_m = \Gamma^{(N-k)}_m \) and all the \( \hat{\theta}_m \) with \( m > 0 \) are degenerated. If for some \( k \), \( \Gamma^{(k)}_m = 0 \), then \( \hat{\theta}_m = 0 \), which implies the system is superstable in the \( m \)th direction.

Strictly speaking, in this situation, any implementation of the system eventually results in superstability in the \( m \)th direction. The synchronization manifold is transversely superstable if, and only if, \( \theta_m = 0 \) in all transversal directions, i.e. for every \( m > 0 \). In average, the time required for a typical trajectory to experience the effects of superstability is of the order of \( \pi_k^{-1} \).

4.2. The effective coupling

Let \( \text{diag}(\lambda_m) \) be a diagonal matrix formed by the effective coupling eigenvalues (8). Let \( P \) be an unitary similarity transformation formed by the Lyapunov vectors of \( S \). Hence, we can make an effective autonomous deterministic system from the definition of an effective coupling matrix:

\[ \tilde{G} = P \text{diag}(\hat{\theta}_m) P^T. \]

From the point of view of the synchronization capacity we show that the systems constructed by \( \{G_n\} \) and \( \tilde{G} \) in (1) are equivalent.

For the systems under consideration, we have \( |G|_{qw} = \exp(-(2\pi i)qw/N) / \sqrt{N} \), therefore

\[ |G|_{iw} = \frac{1}{N} \left\{ \hat{\theta}_0 + 2 \sum_{s=1}^{N} \hat{\theta}_s \cos \left( \frac{2\pi s(w - q)}{N} \right) \right\}, \]  
\[ (9) \]
Once \( \cos(\cdot) \) is an even function with period \( 2\pi \) and \( \delta_0 = 1 \), it is straightforward that \( G \) is a circulant, stochastic and symmetric matrix. For this reason, the effective matrix satisfies all the suppositions about the network. An example is found in Fig. 1.

In general, due to the explicit dependence of the Jacobian matrix on the points of the trajectory, we may formally demonstrate the equivalence between both systems only in the linear neighborhood of \( S \). When such dependence does not exist, the Lyapunov vectors of all points in phase space are identical and our results are globally valid in \( \Omega \) \cite{28,29}. From this, we conjecture that the equivalence between the two systems, for synchronization purposes, is generic. Our results are also valid for networks in which the topology varies periodically in time, replacing \( \pi_k \) by the relative frequency associated with \( k \)th topology.

### 4.3. Example

We shall apply the above results to a system in which, at each time instant, two sites are coupled with probability \( p_a = r^{-2} \); being \( r \) the shortest distance between sites \( 0 \) and \( r, p_b \equiv 1 \), and \( x \geq 0 \) a parameter that quantifies the range of the coupling. We assume the same value of the coupling strength \( \varepsilon \) for all connections. Therefore, the coupling is expressed by

\[
[G_n(T_n)]_{q,w} = \left( [T_n]_{q,w} - (1 + (1 - \varepsilon^{-1})v_n)\delta_{qw} \right) \frac{\varepsilon}{v_n},
\]

in which \( v_n = 2\sum_{r=1}^{N} [T_n]_{r,0} \) is a normalization factor according to (2). Since, by construction, \( p_0 = p_1 = 1 \), the total number of different topologies for this system is \( K_N = 2^{N-1} \).

This example describes a network with time-varying topology and interactions whose probability of coupling between two sites depends on the \( r \) distance between them. This spatial dependence is recovered by the effective matrix \( \hat{G} \) in the form of the coupling intensities, as shown in Fig. 1.

We present an analysis on the synchronization of chaos in this system, as well as illustrations of the equivalence between the original system (1) and effective static system. In view of this, we consider the temporal evolution of the Euclidean distance between the points of the trajectory and \( S \), denoted by \( d_n \equiv \text{dist}(y_n, S) \). As a first examination, we consider the evolution of typical trajectories to \( S \) for a specific realization of this network. The results for the numerical simulations are presented in Fig. 2. From the expressions (8) and (7) we determine a set of parameters \( (N, \varepsilon, x) \) for which \( S \) is transversely stable, in which we chose the tent map \( g(s) = 1 - 2|s - 1/2| (\xi_u = \ln 2) \) to rule the dynamics of each site. The rationale behind this choice is the monotonic convergence of the trajectories to the transversely stable synchronized state, when static couplings are considered \cite{29}. Thus, we can ensure that the irregular time evolution of \( d_n \) in Fig. 2 – for typical trajectories outside the synchronization manifold – is precisely due to the time-dependent coupling. For example, at a given instant \( n \), it is possible that each site is only influenced by a few distant sites, which are in very different states, causing a departure from \( S \). On the other hand, the same mechanism may explain the decrease in the synchronization time when the states are very close together. Fig. 2 also shows the numerical simulations for the static performance of the system, in which we observe that the convergence is monotonic.

We must emphasize that typical trajectories of the system, whose topology is time-varying, present an average behavior that is determined by the equivalent static system, for any set of parameters – including the lattice size – inside the synchronization domain. This result is shown in Fig. 3 in which we analyze the average time required for synchronization of a set of

![Fig. 1. Pictorial representation of \( G \) elements, in which the color scale stands for the coupling intensity. Note the periodicity and translational invariance of the coupling. This matrix has the following parameters: \( N = 31 \), \( \varepsilon = 0.7 \) and \( x = 0.6 \) (elements on the main diagonal are not shown). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)](image-url)
typical trajectories. The synchronization time increases with the parameter \(a\) and diverges at its critical value, \(a_c(\varepsilon, N)\). For this parameter value \(S\) becomes transversely unstable. The \(a_c\) is determined by the Eq. \((8)\) when \(|e_i \bar{\theta}_i| = 1\) \([28]\), being \(\bar{\theta}_i = \max_{m=0}^1(\bar{\theta}_m)\). This critical value \(^3\) is indicated by the dashed lines in Fig. 3. The transition to synchronization, which is characterized by changes in transversal stability of \(S\), occurs at the same point for both systems. In this figure, we also analyze the dispersion of synchronization times. The dispersion is computed by the standard deviation of several network realizations. This quantity is represented by vertical and horizontal bars in the case of systems with varying and static topology, respectively.

5. Conclusions

In conclusion, we studied here the synchronization of chaos in coupled map lattices whose topology varies semi-stochastically with time. Regarding the periodicity, the translational invariance of lattice topology, and that all sites are eventually coupled, we showed that a weighted average over all possible topologies determines the dynamics transverse to the synchronization manifold. This was quantified by the introduction of effective Lyapunov exponents, for which we derive an exact analytical expression. We also showed that, starting from such effective dynamical quantities, a system that is deterministic and autonomous – which we call effective system – can be constructed, so that its topology is static. We demonstrated the equivalence between the two systems by analyzing the stability of the synchronization manifold, and the average time needed for the typical trajectories of the network to reach it. Although the average behavior is the same

---

\(^3\) This network can present superstability at \(\varepsilon = (1 - 1/N) = \varepsilon_{ss}\) for any \(x < \infty\), because global coupling has probability \(\pi_{ss} = (N)!^{-x}\). Typically, this superstability is not observed because the set of parameters, which is determined by \(\varepsilon_{ss}\), has null measure in parameter space. For \(\varepsilon = \varepsilon_{ss} \pm \delta\varepsilon\) the contribution to the expression \((8)\) of the eigenvalue associated with global coupling is given by \(\delta e^{(N)!^{-x}}\), i.e. it becomes evident only for very small networks. For this reason, even for \(\varepsilon = \varepsilon_{ss}\), no manifestation of this phenomenon is observed.
for both systems, we showed that the time dependence of the coupling can cause great variability in time synchronization for the case of isolated observations.

We found an efficient method to numerically determine the effective quantities. This method is based on the eigenvalue decay rate of the coupling matrices’ products, which may be a quick way to calculate the Lyapunov spectrum, because it usually involves a small number of products of matrices (compared to the number of possible topologies). The eigenvalues’ decay is due to the convergence of the coupling matrices product to a matrix of rank 1, because these matrices are stochastic, indecomposable and aperiodic (SIA). SIA matrices appear in different contexts and constitutes a broad class; however, in this work, we use a specific coupling matrix form. Therefore, measuring the rate of decay of the eigenvalues for the determination of effective quantities extends beyond the coupling model used in this work.

In future work, we intend to study the effective quantities for networks with time-varying topology in which the probability of interaction (and not the topology) between elements is invariant over translations. We expect also that the new network considered can be represented by a circulant effective matrix, provided that the network is periodic. If the effective coupling is circulant, the Lyapunov vectors of the synchronization subspace correspond to the eigenvectors of the circulant base matrix and, thus, we can estimate the effective Lyapunov exponents analyzing how close the final shape is each matrix coupling. Consequently, knowing the Lyapunov exponents that characterize the transversal dynamics to the synchronized state, we can find a good estimate for the effective coupling between the elements of the system under study. This is possible with the use of Eq. (9) after inversion of Eq. (7). A well-known example of models whose probability of interaction between elements is invariant over translations are the small-world networks. In these networks one usually assumes that all elements have the same probability of connection. However, few papers have studied the small-world networks with time-varying topology. Therefore, our next step will be to apply the method presented here for this class of systems, and analyze the dependence of the effective quantities on the time scale at which the topology changes.

Acknowledgments

This work has been made possible thanks to the partial financial support from the following Brazilian research agencies: CNPq, CAPES and Fundação Araucária. R.F.P. would like to acknowledge Sinha for presenting the Refs. [10,11] to him at the Dynamics Days South America 2010 coffee break.

References
