

6.8 Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$	s -Shifting (First Shifting Theorem)	6.1
$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$ $\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$ $\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{(n-1)}f(0) - \dots$ $\dots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}\{f\}$	Differentiation of Function Integration of Function	6.2
$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$ $= \int_0^t f(t - \tau)g(\tau) d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5
$\mathcal{L}\{f(t - a) u(t - a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a) u(t - a)$	t -Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(\tilde{s}) d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt$	f Periodic with Period p	6.4 Project 16

6.9 Table of Laplace Transforms

For more extensive tables, see Ref. [A9] in Appendix I.

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \dots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\frac{1}{s-a}$	e^{at}	6.1
8	$\frac{1}{(s-a)^2}$	te^{at}	
9	$\frac{1}{(s-a)^n} \quad (n = 1, 2, \dots)$	$\frac{1}{(n-1)!} t^{n-1} e^{at}$	
10	$\frac{1}{(s-a)^k} \quad (k > 0)$	$\frac{1}{\Gamma(k)} t^{k-1} e^{at}$	
11	$\frac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{(a-b)} (e^{at} - e^{bt})$	6.1
12	$\frac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{(a-b)} (ae^{at} - be^{bt})$	
13	$\frac{1}{s^2 + \omega^2}$	$\frac{1}{\omega} \sin \omega t$	
14	$\frac{s}{s^2 + \omega^2}$	$\cos \omega t$	
15	$\frac{1}{s^2 - a^2}$	$\frac{1}{a} \sinh at$	
16	$\frac{s}{s^2 - a^2}$	$\cosh at$	
17	$\frac{1}{(s-a)^2 + \omega^2}$	$\frac{1}{\omega} e^{at} \sin \omega t$	
18	$\frac{s-a}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$	6.2
19	$\frac{1}{s(s^2 + \omega^2)}$	$\frac{1}{\omega^2} (1 - \cos \omega t)$	
20	$\frac{1}{s^2(s^2 + \omega^2)}$	$\frac{1}{\omega^3} (\omega t - \sin \omega t)$	
21	$\frac{1}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega^3} (\sin \omega t - \omega t \cos \omega t)$	6.6

(continued)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
22	$\frac{s}{(s^2 + \omega^2)^2}$	$\frac{t}{2\omega} \sin \omega t$	} 6.6
23	$\frac{s^2}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t)$	
24	$\frac{s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\frac{1}{b^2 - a^2} (\cos at - \cos bt)$	
25	$\frac{1}{s^4 + 4k^4}$	$\frac{1}{4k^3} (\sin kt \cos kt - \cos kt \sinh kt)$	
26	$\frac{s}{s^4 + 4k^4}$	$\frac{1}{2k^2} \sin kt \sinh kt$	
27	$\frac{1}{s^4 - k^4}$	$\frac{1}{2k^3} (\sinh kt - \sin kt)$	
28	$\frac{s}{s^4 - k^4}$	$\frac{1}{2k^2} (\cosh kt - \cos kt)$	
29	$\sqrt{s-a} - \sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}} (e^{bt} - e^{at})$	5.6
30	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\frac{a-b}{2}t\right)$	
31	$\frac{1}{\sqrt{s^2 + a^2}}$	$J_0(at)$	5.5
32	$\frac{s}{(s-a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}} e^{at}(1 + 2at)$	5.6
33	$\frac{1}{(s^2 - a^2)^k} \quad (k > 0)$	$\frac{\sqrt{\pi}}{\Gamma(k)} \left(\frac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	
34	$e^{-as/s}$	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4
36	$\frac{1}{s} e^{-k/s}$	$J_0(2\sqrt{kt})$	5.5
37	$\frac{1}{\sqrt{s}} e^{-k/s}$	$\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$	
38	$\frac{1}{s^{3/2}} e^{k/s}$	$\frac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$	
40	$\frac{1}{s} \ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	5.6

(continued)

Table of Laplace Transforms (continued)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
41	$\ln \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$	
42	$\ln \frac{s^2 + \omega^2}{s^2}$	$\frac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln \frac{s^2 - a^2}{s^2}$	$\frac{2}{t}(1 - \cosh at)$	
44	$\arctan \frac{\omega}{s}$	$\frac{1}{t} \sin \omega t$	
45	$\frac{1}{s} \operatorname{arccot} s$	$\operatorname{Si}(t)$	App. A3.1

CHAPTER 6 REVIEW QUESTIONS AND PROBLEMS

1. What do we mean by operational calculus?
2. What are the steps needed in solving an ODE by Laplace transform? What is the subsidiary equation?
3. The Laplace transform is a linear operation. What does this mean? Why is it important?
4. For what problems is the Laplace transform preferable over the usual method? Explain.
5. What are the unit step and Dirac's delta functions? Give examples.
6. What is the difference between the two shifting theorems? When do they apply?
7. Is $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$? Explain.
8. Can a discontinuous function have a Laplace transform? Does every continuous function have a Laplace transform? Give reasons.
9. State the transforms of a few simple functions from memory.
10. If two different continuous functions have transforms, the latter are different. Why is this practically important?

11–22 LAPLACE TRANSFORMS

Find the transform (showing the details of your work and indicating the method or formula you are using):

11. te^{3t} 12. $e^{-t} \sin 2t$

13. $\sin^2 t$ 14. $\cos^2 4t$
15. $tu(t - \pi)$ 16. $u(t - 2\pi) \sin t$
17. $e^t * \cos 2t$ 18. $(\sin \omega t) * (\cos \omega t)$
19. $\sin t + \sinh t$ 20. $\cosh t - \cos t$
21. $e^{at} - e^{bt} \quad (a \neq b)$ 22. $\cosh 2t - \cosh t$

23–34 INVERSE LAPLACE TRANSFORMS

Find the inverse transform (showing the details of your work and indicating the method or formula used):

23. $\frac{10s}{s^2 + 2}$ 24. $\frac{15}{s^2 - 4}$
25. $\frac{12}{s^2 + 4s + 20}$ 26. $\frac{3s}{s^2 - 2s + 2}$
27. $\frac{5s + 4}{s^2} e^{-2s}$ 28. $\frac{2s - 10}{s^3} e^{-5s}$
29. $\frac{2s + 4}{(s^2 + 4s + 5)^2}$ 30. $\frac{s^2 - 16}{(s^2 + 16)^2}$
31. $\left(\frac{2}{s^2} + \frac{2}{s^3}\right) e^{-s}$ 32. $\frac{180 + 18s^2 + 3s^4}{s^7}$
33. $\frac{\pi}{s^2(s^2 + \omega^2)}$ 34. $\frac{2}{2s^2 + 2s + 1}$