A New Distortion Model for Strong Inhomogeneity Problems in Echo-Planar MRI

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Abstract—This work proposes a new distortion model for strong inhomogeneity problems in Echo Planar Imaging (EPI). Fast imaging sequences in MRI, such as EPI, are very important in applications where temporal resolution or short total acquisition time is essential. Unfortunately, fast imaging sequences are very sensitive to variations in the homogeneity of the main magnetic field. The inhomogeneity leads to geometrical distortions and intensity changes in the image reconstructed via Fast Fourier Transform. Also, under strong inhomogeneity, the accelerated intravoxel dephase may overly attenuate signals coming from regions with higher inhomogeneity variations. Moreover, coarse discretization schemes for the inhomogeneity are not able to cope with this problem, producing discretization artifacts when large inhomogeneity variations occur. Most of the existing models do not attempt to solve this problem. In this paper, we propose a modification of the discrete distortion model to incorporate the effects of the intravoxel inhomogeneity and to minimize the discretization artifacts. As a result, these problems are significantly reduced. Extensive experiments are shown to demonstrate the achieved improvements. Also, the performance of the new model is evaluated for conjugate phase, least squares method (minimized iteratively using conjugated gradients), and regularized methods (using a total variation penalty).

Index Terms—magnetic resonance imaging, echo-planar imaging, magnetic field inhomogeneity, discretization, regularization, total variation.

I. INTRODUCTION

Magnetic resonance imaging (MRI) is an important tool for medical diagnostics. The standard algorithm used for image reconstruction is the inverse Fourier transform [1], since the acquired data are in the Fourier space, known as k-space. In the conventional imaging sequences, i.e. spin echo or gradient echo, a large number of radio-frequency (RF) excitations are needed in order to capture enough data to reconstruct an image. Due to the long period required between RF excitations, the complete acquisition procedure may take too much time. Besides being uncomfortable for the patient and more susceptible to motion artifacts, the long time reduces temporal resolution provided by the imaging sequence, which is important for several applications. Because of this, fast imaging sequences become very important where temporal resolution is essential, such as in functional MRI (fMRI) [2] and cardiac imaging [3], or where total acquisition time must be short, such as in diffusion tensor imaging [4], [5]. Fast imaging sequences, like echo-planar imaging (EPI) and spiral imaging, are able to acquire the entire k-space data with only one RF excitation, considerably reducing the acquisition time.

Even though fast imaging sequences are able to capture more data from the RF excitation [6], they are subject to longer readout periods and more susceptible to imperfections of the acquisition system. As soon as the RF excitation is over, the signal emitted by the object begins naturally to reduce its strength and to accumulate phase errors [1]. These fast imaging sequences are usually very sensitive to variations in the homogeneity of the main magnetic field and also require stronger and fast switching magnetic gradients, sometimes forcing the magnetic gradients system to the limit of the MRI scanner [7]. These imperfections of the system may corrupt the captured data [1], [6].

One of the main inconveniences, which is treated in this paper, is the magnetic field inhomogeneity caused by the main field imperfections and by differences in magnetic susceptibility in air/tissue interface [6]. In EPI, the inhomogeneity leads to geometrical distortions and intensity variations, while in spiral imaging it causes spatial blurring and artifacts. Besides the geometrical distortions, blurring and artifacts, which are macro effects of the inhomogeneity acting during the readout period, there are some micro effects caused by the intravoxel inhomogeneity. The intravoxel inhomogeneity causes the MR signal to lose its strength faster, due to the cancellation of the magnetic moment of the spins with different phases within a particular voxel [1].

In spiral imaging, data are captured in a non-cartesian grid, either requiring a gridding procedure [8] previously to the application of the Fast Fourier transform (FFT) or the use of the non-uniform-FFT (NUFFT) [9], [10], even if the inhomogeneity problem is not considered. On the other hand, EPI data are captured in a rectangular grid, making reconstruction easier. Also, in EPI, the inhomogeneity causes more severe distortions in the phase encoding direction, while distortions in the frequency encoding direction are almost imperceptible in most applications [11], [12], [13]. This allows approximating the two-dimensional (2D) acquisition problem by several small one-dimensional (1D) problems, making the analysis and the understanding of the problem much simpler. In this paper, we consider the EPI acquisitions due to this simplicity, even though the results can be extended to other fast imaging techniques as spiral imaging.

Several methods have been proposed to correct the inhomogeneity problem. The most frequently used ones are conjugate phase (CP) methods [14], [15] and iterative methods [11], [12], [16]. Among them, the linear conjugate gradient (CG) method [17] is the most common. When the inhomogeneity is not too severe, the conjugate phase produces very satisfactory results. As the inhomogeneity increases, some intensity changes appear, then iterative methods that are able to correct the intensity can be used, as the square norm minimized with conjugated gradient [11], [12], [16]. However, when the inhomogeneity becomes stronger, the direct problem becomes more ill-posed and regularization is needed [17], [18]. Above a certain limit, the problem gets close to singular, and the use of prior information is required in order to get satisfactory results.

The use of prior information about the image is a necessity in inverse image problems, like computer tomography [19], positron emission tomography [20], image restoration [19], super-resolution [21], [22], [23], the emminent compressive sampling theory [24], [25], among others. A common form of inclusion of prior information in ill-posed problems is through the use of an additive regularizing.

Manuscript received February 11, 2009; revised April 02, 2009.

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Work by the first author was supported by FAPESP grant number 06/06797-4. and the second author by FAPESP grant No 2002/07153-2 and CNPq grants No 476825/2006-0 and 304820/2006-7

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penalty [17], [26]. This form is directly connected with the statistical maximum a posteriori (MAP) estimation [19], [27], where a priori distribution of the solution is naturally included in the estimation process. One of the penalties that has been shown very efficient for imaging problems is the total variation [17]. This regularizing penalty helps to recover smooth images, but preserving boundary discontinuities, reconstructing images with high contrast, and very little visible artifact. This kind of regularization has been proven to be adequate for a wide class of images [19].

Under strong magnetic inhomogeneity, the distortion problems, such as geometrical distortions and intensity variations, as well as intravoxel signal attenuation must be considered for a proper restoration. Most of the current work do not attempt to solve this joint problem, some exceptions are [28] and [29]. In [28], the intravoxel dephasing is considered in spiral acquisitions and is assumed to cause a signal attenuation that follows an exponential decay along time. This is a valid model when the frequencies of the spins inside the voxels follow a Lorentzian distribution [1]. In [29], the shape of the signal attenuation along time is generalized to an arbitrary form in EPI. However, the article [29] considers only the intravoxel signal attenuation caused by inhomogeneity in the slice direction, but not the intravoxel inhomogeneity in the in plane directions, perpendicular to the slice selection direction.

Besides signal attenuation and distortion problems, the discretization scheme also affects the performance of the discrete reconstruction methods, especially when strong geometrical distortions occur in EPI. The discretization scheme commonly used, assumed in [11], [12], [29], often generates artifacts in the resulting images. We propose a modification of the discrete distortion model to incorporate the effects of the intravoxel inhomogeneity acting in the in plane directions, as well as the discretization artifacts. As a result, these effects can be significantly reduced.

In this paper, we propose a new model for the distortion introduced by the inhomogeneity, which includes intravoxels inhomogeneity and discretization problems. The model is based on considering a continuous approach for the inhomogeneity field map, allowing higher resolution levels for better describing intravoxel behavior. This is crucial in the presence of strong inhomogeneities. The proposed model deals better with heavy geometric distortions and intensity variations than the current model [11], [12]. Our objective is to show that it is possible to obtain satisfactory reconstructions in EPI even under very strong inhomogeneity distortions, which could easily make the image unusable.

In Section II the EPI problem is presented, introducing the standard discrete model as well as the proposed one. Section III reviews the typical correction methods that can make use of our model, including regularization with Total Variation penalties. In Section IV the results of an extensive evaluation are shown, including the effect of different levels of resolution derived from our model compared to the current model. Finally, in Section V, a discussion about the problem is presented and conclusions are drawn.

II. THEORY

The equation that establishes the relation between the magnetization of the voxels and the MR signal is:

\[ s(t) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi [x^T k(t)]} dx \]

(1)

Where \( s(t) \) is the MR signal, \( f(x) \) is the image voxel at the 2D position \( x = [x, y]^T \), or 3D position \( x = [x, y, z]^T \) and \( k(t) \) is the k-space position (2D or 3D) at time \( t \) after RF excitation. For this paper, we assume \( z \) as the slice position of the 2D images. Functions with parenthesis, "()", as \( s(t) \) and \( f(x) \), are assumed to be continuous functions. The k-space position at time \( t \) depends on the action of the magnetic gradients, coordinated by the imaging sequence, given by:

\[ k(t) = \frac{\gamma}{2\pi} \int_{0}^{t} G(\tau)d\tau \]

(2)

Where \( G(t) = [G_x(t), G_y(t)]^T \) are the gradient signals of the imaging sequence and \( \gamma \) is the gyromagnetic ratio of protons (\( \gamma/2\pi = 42.6 \text{ MHz/T} \)).

In the rectilinear 2D trajectory echo-planar imaging [1], [6], the signaling sequence is given in Figure 1(a). This sequence leads to the k-space trajectory shown in 1(b). In this sequence, the \( G_x(t) \) gradient provides the selection of the slice to be imaged.

![Fig. 1. EPI sequence and k-space trajectory.](image)

A. Imperfections in the MR Acquisition

Equation (1) relies on a perfect acquisition system. However, difficulties may arise when the main magnetic field is not homogeneous and the response of the magnetic gradient is not ideal. In those cases, the space-frequency mapping is no longer linear. The following equation provides a more complete representation of the acquisition with imperfections in the MR system:

\[ s(t) = \int_{-\infty}^{\infty} M(x) e^{-i2\pi [x^T k(t) + \Delta k(t)] + \frac{\gamma}{2\pi} \Delta b(x) t} \]

\[ \times e^{-\frac{t}{T_2(x)}} dx + \eta(t) \]

(3)

In (3) it is included the attenuation caused by the \( T_2 \) process, the deviation from the expected k-space position in \( \Delta k(t) \), the frequency-space deviation caused by the magnetic field inhomogeneity in \( \Delta b(x) \) and noise, represented by \( \eta(t) \), which is assumed to be white complex Gaussian (WCG) noise\(^1\) and follow the same Gaussian distribution. [27]. The \( M(x) \) is the transverse magnetization.

The exponential function of the \( T_2 \) process leads to small filtering artifacts, which is usually ignored in practice, and to weighting effects in the transverse magnetization, which is sometimes useful for medical diagnostics. The k-space deviation, modeled by \( \Delta k(t) \), is the main cause of the N/2 ghosting in EPI images [6], [30]. When this error is well structured it can be corrected after the acquisition [30], [31], [32], [33] with relatively good results. Unconsidering the filtering caused by \( T_2 \) process and assuming that the N/2 ghosting was properly corrected, the remaining imperfections from (3) are the inhomogeneity and the noise:

\[ s(t) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi [x^T k(t) + \frac{\gamma}{2\pi} \Delta b(x) t]} \]

\[ \times e^{-\frac{t}{T_2(x)}} dx + \eta(t) \]

(4)

\(^1\)White complex Gaussian noise is a complex random process where each element, or point in time, is independent, its real and imaginary components are also independent.
where $f(x) = M(x)e^{-\frac{tx}{T2s}}$ is the $T2$ weighted transverse magnetization and $T2c$ is the echo time, assumed to be the time when the center of the k-space is captured. In equation (4), $t$ specifies the time after the RF excitation pulse. Observe also, that there are no intravoxel effects since we are dealing with continuous functions.

### B. Previous Discrete EPI Equation

According to the discretization model assumed in [11, 12], after sampling the signal in the k-space and assuming a discrete format for the target image, equation (4) becomes:

$$s[k_x, k_y] = \sum_{m_x=0}^{M_x-1} \sum_{m_y=0}^{M_y-1} f[m_x, m_y] B[k_x, m_x; k_y, m_y] + \eta[k_x, k_y]$$

(5)

with

$$B[k_x, m_x; k_y, m_y] = e^{-i2\pi \left[ \frac{k_x m_x}{M_x} + \frac{k_y m_y}{M_y} + \frac{\gamma}{2\pi} \Delta b[m_x, m_y] t[k_x, k_y] \right]}$$

(6)

where $m_x$ and $m_y$ are the vertical and horizontal discrete spatial variables of an image $f$, respectively, with size $M_x \times M_y$. $k_x$ and $k_y$ are the k-space variables, $\Delta b[m_x, m_y]$ is the discrete magnetic inhomogeneity field, and $t[k_x, k_y]$ is the time between sampling the k-space point and the reconstruction point. In this paper, brackets, “[]”, as in $\Delta b[m_x, m_y]$, are used to represent discrete functions.

The system of equations (5) can be expressed in the matrix-vector form, as:

$$s = Bf + \eta$$

(7)

with the elements of the data vector of size $M \times 1$, where $M = M_x M_y$, being $s[k_x, k_y] = s[k_x, k_y]$, the elements of the image being $f[m_x, m_y]$, and the elements of the matrix $B[k_x, m_x; k_y, m_y] = B[k_x, m_x; k_y, m_y]$. The vectors and the matrix in (7) are all complex valued. Note that the vector $s$ is a stack of the columns of the k-space data. We use italic bold symbols to represent the systems with dimension $M \times M$.

The system in (7) is too large to be used in practice for analysis. This approximation is described next.

1) From the 2D Inhomogeneity Problem to the 1D: For most practical situations, it is possible to assume that the time $t(k_x, k_y)$ depends only on the phase encoding direction, assumed in this paper as the $k_x$ axis, or vertical axis, as shown in Figure 1(b).

$$B[k_x, m_x; k_y, m_y] = e^{-i2\pi \left[ \frac{k_x m_x}{M_x} + \frac{k_y m_y}{M_y} + \frac{\gamma}{2\pi} \Delta b[m_x, m_y] t[k_x, k_y] \right]}$$

(8)

Under this assumption, it is possible to apply the inverse Fourier transform (IFT) in the $k_y$ axis of (5) (the frequency encoding axis, assumed to be horizontal axis in this paper). As a result we have the following $M_y$ small 1D approximating problems [12]:

$$s[k_x, n_y] = \sum_{k_y = -M_y/2}^{M_y/2-1} s[k_x, k_y] e^{i2\pi \frac{k_y n_y}{M_y}}$$

(9)

$$s[k_x, n_y] = \sum_{m_y = 0}^{M_y-1} f[m_x, n_y] B[k_x, m_x; n_y] + \eta[k_x, n_y]$$

See equation (35) in the appendix for the detailed steps to reach the equalities in (9).

The system of equations (9) can be expressed in the matrix-vector form, considering $n = n_y$, as:

$$\bar{s}_n = \bar{B}_n f_n + \bar{\eta}_n$$

(10)

with the elements of the data vector of size $M_x \times 1$ being $s[k_x, n] = \bar{s}[k_x, n]$, the elements of the image $f[n_x, n] = f[m_x, n]$, and the elements of the matrix $\bar{B}_n[k_x, m_x] = \exp \left( -i2\pi \frac{k_x m_x}{M_x} \right)$. We use only the bold symbols to represent systems with dimension $M_x \times M_x$, and the bar, as in $\bar{s}$, to represent the systems 1D inverse Fourier transformed in the frequency encoding axis.

2) Sparse Distortion Matrix: The reconstruction of the image by applying the 2D IFT in the distorted signal from (5), or the 1D IFT in (10), leads to a distorted image, given by:

$$f'_n = \mathbf{F}^{-1} \bar{B}_n f_n + \mathbf{F}^{-1} \bar{\eta}_n = \mathbf{H}_n f_n + \eta'_n$$

(11)

where $F^{-1}$ is the 1D IFT applied to the phase encoding axis, and $\eta'_n$ is the vector with the $n$-th vertical line of the distorted EPI image.

The matrix $\mathbf{H}_n$, in (11), is the 1D distortion matrix [11, 12]. The advantage of using this system is that it is very sparse, requiring less storage space than the matrix $\bar{B}_n$. Besides, the picture of the absolute values of the elements of this matrix gives us an insight of how the distortion caused by the inhomogeneity will act. An example is shown in Figure 2. In the absence of inhomogeneity this matrix is equal to the identity, when the inhomogeneity is between weak to medium, it is very close to the identity, as in Figure 2(a). As the inhomogeneity increases it becomes more off diagonal, as in Figure 2(b).

![Weak medium inhomogeneity](image1.png)

(a) Weak medium inhomogeneity

![Strong inhomogeneity](image2.png)

(b) Strong inhomogeneity

Fig. 2. A picture of the absolute values of the elements of the distortion matrix.

### C. Proposed Discrete EPI Equation

The discretization of the acquisition matrix in (6) is overly simplified and does not consider the intravoxel inhomogeneity. This discretization also causes some artifacts when the inhomogeneity is strong, as will become clear in forthcoming sections.

In order to provide a more faithful discretization we assume that the target image can be represented in the continuous form as:

$$f(x) = f(x, y) = \sum_{m_x=0}^{M_x-1} \sum_{m_y=0}^{M_y-1} f[m_x, m_y] \phi_x(x-m_x) \phi_y(y-m_y)$$

(12)

where $f[m_x, m_y]$ is the discrete image, $f(x, y)$ is the continuous image, and $\phi_x(x-m_x)$ and $\phi_y(y-m_y)$ are the continuous interpolation functions. For band-limited periodic images, $\phi$ is the periodic sinc function, also known as the Dirichlet kernel [1], [34], [35], defined by:

$$\text{diric}(x, n) = \sum_{k=-n}^{n} e^{i2\pi k x/M} = \frac{\sin(\pi x/M(n/2+1))}{\sin(\pi x/M)}$$

(13)
which is periodic for each \( x \) multiple of \( M \), while \( n \) is the maximum frequency of the function.

Substituting (12) back in (4), and assuming the signal \( s(t) \) to be regularly sampled [1], we have the following discrete equation:

\[
s[k_x, k_y] = \sum_{m_x=0}^{M_x-1} \sum_{m_y=0}^{M_y-1} f[m_x, m_y] A[k_x, m_x; k_y, m_y] + \eta[k_x, k_y]
\]  
(14)

which equals to (5), except that the acquisition system is now:

\[
A[k_x, m_x; k_y, m_y] = \int_0^{M_x} \int_0^{M_y} \phi_x(x - m_x) \phi_y(y - m_y) \times e^{j \frac{k_x x}{M_x} + \frac{k_y y}{M_y} + \frac{\gamma}{2\pi} b(x, y)t[k_x, k_y]} dx dy
\]  
(15)

Note that it is necessary to solve the integrals in (15) to obtain each element of the discrete matrix. Also, one needs to obtain the field map at the continuous level, which is not available in general. The matrix-vector form of (14) is the same as in (7) but the matrix is different.

1) Resolution Level of the Discrete System: One practical way to obtain (15) is to simplify the integrals and calculate it numerically as sums. However, in order to accurately approximate the integrals, more samples than the ones of the original \( M_x \times M_y \) grid are needed. Assuming a grid with \( R \) times more samples, then (15) is approximated by:

\[
A[k_x, m_x; k_y, m_y] = \frac{1}{R^2} \sum_{x=0}^{RM_x-1} \sum_{y=0}^{RM_y-1} \phi_x(x - Rm_x) \phi_y(y - Rm_y) \times e^{j \frac{k_x x}{RM_x} + \frac{k_y y}{RM_y} + \frac{\gamma}{2\pi} b(x, y)t[k_x, k_y]} dB
\]  
(16)

In (16), \( R \) is defined as the resolution level of the discrete matrix. Observe that, \( \phi_x(x) \) and \( \phi_y(y) \) are the sampled values of \( \phi_x(x) \) and \( \phi_y(y) \) respectively. Also, since these functions are interpolation functions, if \( R = 1 \), then \( \phi_x(x - Rm_x) \) and \( \phi_y(y - Rm_y) \) should be discrete delta functions [34, 35] and (16) should be equal to (6). Also, observe that the field map needs to have \( R^2 \) times more samples than the image \( f \), which makes sense since we want to consider the intravoxel inhomogeneity effects. If a high-resolution field map is available, one can use an interpolated version of the low-resolution field map, however, it may not reproduce precisely the intravoxel inhomogeneity. In Section IV we will offer an evaluation of different values of \( R \) to observe how many extra samples are necessary to compute the matrix \( A \).

Optionally, one may choose to use different resolution levels in each direction, considering \( R_x \) and \( R_y \) as the resolution levels in the \( x \) and \( y \) axis, respectively. This form may present some advantages regarding the computational cost, as will be seen next.

2) From 2D to 1D with Intravoxel Inhomogeneity: As for (6), the matrix constructed from (16) is too dense to be used in practice. The approximation assumed in (8), which considers that the time depends only on \( k_x \), is also valid here. Then, applying the 1D IFT in (14), we get:

\[
\tilde{s}[k_x, n_y] = \sum_{k_y} \sum_{n_y} s[k_x, k_y] e^{j \frac{2\pi}{M_y} k_x n_y}
\]
and

\[
\tilde{s}[k_x, n_y] = \sum_{m_x=0}^{M_x-1} \sum_{m_y=0}^{M_y-1} f[m_x, m_y] \tilde{A}[k_x, m_x; n_y, m_y] + \tilde{\eta}[k_x, n_y]
\]  
(17)

by expanding \( \tilde{A}[k_x, m_x; n_y, m_y] \):

\[
\tilde{A}[k_x, m_x; n_y, m_y] = \frac{1}{R^2} \sum_{x=0}^{RM_x-1} \sum_{y=0}^{RM_y-1} \phi_x(x - Rm_x) \phi_y(y - Rm_y) \times e^{j \frac{k_x x}{RM_x} + \frac{k_y y}{RM_y} + \frac{\gamma}{2\pi} b(x, y)t[k_x, k_y]} dB
\]  
(18)

where \( \phi_y(y - Rm_y) \) is a discrete version of the periodic sinc function, shown in equation (13), shifted by \( Rm_y \). The complete development of equations (17) and (18) are shown in the appendix, in equations (36) and (37), respectively.

It is important to notice that the results of a particular output line \( n_y \) vertical line, in this paper) no longer depend only on the same index input line, i.e., when \( n_y = m_y \), except if \( R = 1 \) (or optionally, if \( R_0 = 1 \)). This means that neighboring voxels also influence the results depending on how the product \( \phi_x(x - Rm_x) \phi_y(y - Rm_y) \) performs as the difference between \( n_y \) and \( m_y \) increases. Then, assuming \( n = n_y \) and \( m = m_y \), we have:

\[
\tilde{s}_n = \sum_{m=1}^{M_y} \tilde{A}_{n,m} \bar{r}_m + \tilde{\eta}_n
\]  
(19)

with the elements of the data vector of size \( M_x \times 1 \) being \( [\tilde{s}_n] = \tilde{s}[k_x, n] \), the elements of the image \( [\bar{f}_m] = f[m_x, m_y] \), and the elements of the matrix \( [\tilde{A}_{n,m}] = [\tilde{A}[k_x, m_x; n_y, m_y]] \).

We expect the matrices \( \tilde{A}_{n,m} \) to become near zero matrices as the difference \( |n - m| \) increases. If the matrix norm of the \( \tilde{A}_{n,m} \) matrices vanishes quickly as \( |n - m| \) increases, then it is not necessary the use of the matrices with large \( |n - m| \).

3) Sparse Distortion Matrix: We can apply the 1D IFT along the \( k_x \) axis in (17), in order to recover the distortion system, shown as:

\[
f'[n_x, n_y] = \sum_{m_x=0}^{M_x-1} \sum_{m_y=0}^{M_y-1} f[m_x, m_y] H(n_x, m_x; n_y, m_y) + \eta'[n_x, n_y]
\]  
(20)

where

\[
H(n_x, m_x; n_y, m_y) = \sum_{k_x = -\frac{M_x}{2}}^{\frac{M_x}{2} - 1} \tilde{A}[k_x, m_x; n_y, m_y] e^{j \frac{2\pi}{M_x} k_x n_x}
\]
(21)

for

\[
\tilde{H}[n_x, m_x; y] = \sum_{x=0}^{RM_x-1} \phi_x(x - Rm_x) \times e^{j \frac{2\pi}{2\pi} b(x, y)t[k_x, k_y]} dB
\]  
(22)
The complete development of equations (20), (21) and (22) are in the appendix, detailed in equations (38) and (39).

Observe that in (21), for each \( y \), the system in \( \tilde{H}[n_x, m_x; y] \) remains an independent \( m_x \) to \( n_x \) mapping. However, the sum over \( y \), under the influence of \( \phi_y[y - R[R_y]] \), will include neighboring voxels in the equation.

The full discrete system can be represented in the following form:

\[
\mathbf{f}' = \mathbf{H} \mathbf{f} + \boldsymbol{\eta}'
\]  
(23)

where

\[
\mathbf{H} = 
\begin{bmatrix}
H_{1,1} & H_{1,2} & \cdots & H_{1,m_y}

H_{2,1} & H_{2,2} & \cdots & H_{2,m_y}

\vdots & \vdots & \ddots & \vdots

H_{m_y,1} & H_{m_y,2} & \cdots & H_{m_y,m_y}
\end{bmatrix}
\]  
(24)

and

\[
\mathbf{f}' = \begin{bmatrix} f'_1^T f'_2^T \cdots f'_{m_y}^T \end{bmatrix}^T, \quad \mathbf{f} = \begin{bmatrix} f_1^T f_2^T \cdots f_{m_y}^T \end{bmatrix}^T, \quad \boldsymbol{\eta}' = [\eta_1' \eta_2' \cdots \eta_{m_y}']^T, \quad \text{and} \quad [H_{n_x, m_y}(n_x, m_y)] = H[n_x, m_x; n_y, m_y].
\]

It is worth noting that the vectors and matrices in (24) are all complex-valued.

If ones decides to use independent resolution factors \( R_n \) and \( R_p \), and chooses \( R_q = 1 \), then the matrix \( \mathbf{H} \) in (24) becomes a block diagonal matrix, which can be solved independently, as in (11). In this case the matrices \( \mathbf{H}_n \) in (11) correspond to the block matrices \( \mathbf{H}_{n,n} \) in (24), while the side blocks, \( \mathbf{H}_{n,m} \) with \( n \neq m \), are all zero blocks.

If \( R_q > 1 \), it still worth noting that each matrix \( \mathbf{H}_{n,m} \) is also sparse, and has few non-zero elements. Also, since we expect the norm of the matrix \( \mathbf{H}_{n,m} \) to vanish quickly as the difference \( |n - m| \) increases, we will actually have few non-zero side blocks. Figure 3 shows graphics of the absolute values of the elements of the matrices \( \mathbf{H}_{n,n} \) and \( \mathbf{H}_{n,n+1} \). Observe that the components of an off-diagonal block matrix have much lower absolute values than the components of a diagonal block matrix.

(a) Matrix \( \mathbf{H}_{n,n} \) under strong inhomogeneity (b) Matrix \( \mathbf{H}_{n,n+1} \) under strong inhomogeneity

Fig. 3. Absolute values of the elements of the distortion matrix \( \mathbf{H}_{n,n} \) and \( \mathbf{H}_{n,n+1} \).

4) Approximation for the Sparse Distortion Matrix: Since the matrix \( \mathbf{H} \) in (24) is composed by sparse blocks tending to zero as the difference \( |n - m| \) increases, one can construct an approximate matrix, completely removing the blocks far from the block diagonal. As we will demonstrate in section IV, this approximation is very useful, because it provides much better quality than the single block diagonal system and has much lower computational cost than the full system with all the blocks.

Therefore, the approximate discrete system can be represented in the following form:

\[
\mathbf{f}' = \tilde{\mathbf{H}} \mathbf{S} \mathbf{f} + \boldsymbol{\eta}'
\]  
(25)

where:

\[
\tilde{\mathbf{H}}_S = 
\begin{bmatrix}
H_{1,1} & \cdots & H_{1,S+1} & 0 & \cdots & 0

\vdots & \ddots & \vdots & \ddots & \ddots & \vdots

H_{S+1,1} & \cdots & H_{S+1,S+1} & \cdots & H_{S+1,2S+1} & 0

0 & \cdots & 0 & \cdots & 0 & \cdots

\vdots & \ddots & \vdots & \ddots & \ddots & \ddots

0 & \cdots & 0 & \cdots & \cdots & \cdots & H_{M_y,S,M_y - S} & \cdots & H_{M_y,S,M_y}
\end{bmatrix}
\]  
(26)

\( S \) is the number of blocks off the main block diagonal to the left and to the right, in a total of \( 2S \) side blocks. We expect \( S \) to be small in order to keep the computational cost at levels close to the system with only the main block diagonal.

III. METHODS

The most commonly used methods to correct for inhomogeneities are the conjugate phase methods [14], [15] and the least squares methods [11], [12]. However, due to the ill-posedness of the system with strong inhomogeneities, we expect the regularized least squares to perform better.

A. Conjugate Phase Method

The conjugate phase method is perhaps the most widely used method to correct for inhomogeneities. According to [12], it is described by the following equation for EPI problems:

\[
\hat{f}_{cp} = \mathbf{H}^* \mathbf{f}', \quad n = 1 \ldots M_y
\]  
(27)

where \( \mathbf{H}^* \) is the complex conjugate transpose of \( \mathbf{H} \), given in (11). For this method, usually the 1D system with standard discretization scheme is adopted, however it is possible to use the 2D system as well [12]. The conjugate phase method with the 2D system is:

\[
\hat{f}_{cp} = \mathbf{H}^* \mathbf{f}'
\]  
(28)

where \( \mathbf{H}^* \) is the complex conjugate transpose of \( \mathbf{H} \), which is given in (24), or its approximation from (26), if our proposed discretization is chosen. If the current discretization is adopted, then \( \mathbf{H} \) is equivalent to (26), with \( \mathbf{S} = 0 \), and \( \mathbf{H}_{n,n} = \mathbf{H}_n \).

B. Least Squares Methods

The least squares method is one of the most widely used method in various types of inverse problems [18], [17]. Using the reduced 1D model, it is described by:

\[
\hat{f}_{ls}^{\mathbf{S}} = \arg \min_{f_n} \| f_n - \mathbf{H}_n f_n \|_2^2, \quad n = 1 \ldots M_y
\]  
(29)

The solution of (29) is:

\[
\hat{f}_{ls}^{\mathbf{S}} = (\mathbf{H}_n^T \mathbf{H}_n)^{-1} \mathbf{H}_n^T f_n, \quad n = 1 \ldots M_y
\]  
(30)

If the inhomogeneity problem is not significantly strong, the matrix \( \mathbf{H}_n^T \mathbf{H}_n \) is close to the identity, and this method almost coincides with conjugate phase. If this is not the case, inversion as in (30) is necessary for a better correction. For strong inhomogeneity problems the matrix \( \mathbf{H}_n \) becomes very ill-conditioned, close to singular. Then, the use of a regularized inverse becomes mandatory. In [11] the direct
solution via truncated singular value decomposition and the iterative solution via conjugate gradient [17] method are proposed. The least squares solution can also be implemented with the 2D system, expressed as:

$$f^{ls} = \arg \min_{f} ||f - Hf||^2_2$$  \hspace{1cm} (31)

with the solution:

$$f^{ls} = (H^*H)^{-1}H^*f'$$  \hspace{1cm} (32)

where $H$ is the 2D distortion, in which one can use the proposed discretization scheme in (24), or its approximation in (26). A more interesting class of regularized solutions arises when using an additive penalty based on prior information about the solutions.

C. Regularized Least Squares Methods

The regularized least squares has the following form in the 2D problem:

$$f^{rls} = \arg \min_{f} ||f - Hf||^2_2 + \lambda R(f)$$  \hspace{1cm} (33)

where $R(f)$ is the 2D regularization penalty and $\lambda$ is the regularization parameter, which controls the influence of the regularization term in the solution. The 1D problem can be easily derived, similarly to (29).

The formulation in (33) is strongly connected with maximum posteriori (MAP) estimation, where the regularization penalty is specified by the prior distribution of the solution. This method has been successfully applied for solving different imaging problems in computer tomography [19], positron emission tomography [20], image restoration [19] and super-resolution [21], [22], [23].

Typical regularization terms found in imaging problems are quadratic penalties, where $R(f) = ||Rf||^2_2$, as in [19], [17]. The matrix $R$ is usually a discrete derivative operator. The most common matrices $R$ are derived from first or second order finite differences. Another important regularization term in imaging literature is the total variation penalty [17]:

$$R(f) = TV(f) = \sum_{x,y} \sum_{m_x = 2m_y = 2} \left( |f(m_x, m_y) - f(m_x - 1, m_y)|^2 + |f(m_x, m_y) - f(m_x, m_y - 1)|^2 \right)^{1/2}$$  \hspace{1cm} (34)

This penalty produces images with better-defined edges, providing significant improvement. Moreover, this kind of penalty is connected with successful results obtained in compressive sampling theory [24], [25]. In our experiments TV will be our choice as penalty function.

IV. RESULTS

A. Simulations to Evaluate the Discretization Scheme

The objective of this section is to evaluate our proposed discretization scheme. One of the motivations for this new model is given by the results in Figure 4, where the inhomogeneity is strong and causes significant distortions in the EPI image. The effect of the inhomogeneity can be observed by comparing the captured gradient recalled echo (GRE) image, shown in Figure 4(a), with the captured EPI image, shown in Figure 4(c). Geometrical distortions and localized signal attenuations are quite visible in the EPI. We want to observe the behavior of the singular values when varying the magnitude of the inhomogeneity, in order to observe ill-posedness; 2) the influence of the resolution level of the discrete system, and 3) the necessary numbers of side blocks in the approximate distortion matrix. The experiments were also repeated for other artificial images and field maps, some of them shown in Figure 17.

1) Evaluating Singular Values Varying the Magnitude of the Inhomogeneity: We want to observe the behavior of the singular values of the matrix $H_m$ as the inhomogeneity increases. In Figure 4(b), positive values for the field map cause the pixels in the EPI image to shift down, and negative values to shift up. In Figure 4(d), that shows the vertical derivative of the field map, one can see that a negative derivative leads to a compression of the pixels in the EPI image, while a positive derivative leads to an expansion. In [14] it was demonstrated that if $\partial (\gamma \Delta h(x, y)/G_x)/\partial x \leq -1$, in a continuous...

When we try to artificially reproduce the EPI image, using the GRE image and the estimated field map, with the current discretization model, as shown in Figure 4(e), we observe that the resulting EPI image was not very similar to the true EPI image. The intravoxel signal attenuation was not well reproduced, specially in high horizontal variations of the field map, as in the detail in Figure 4(h), and discretization artifacts appeared, mainly in the expanding regions, as detailed in Figure 4(g). However, considering the GRE image with higher resolution ($R$ times more) in the simulated acquisition, we observe much better results, as shown in Figure 4(f). These results are without discretization artifacts and with some visible intravoxel attenuation. The experiments above lead us to the new model. In the following we evaluate three aspects of the new model: 1) the behavior of the singular values when varying the magnitude of the inhomogeneity, in order to observe ill-posedness; 2) the influence of the resolution level of the discrete system, and 3) the necessary numbers of side blocks in the approximate distortion matrix. The experiments were also repeated for other artificial images and field maps, some of them shown in Figure 17.
EPI image, then, that region is singular. Which is equivalent to say, in the discrete sense, that if the discrete vertical derivative of the field map, as shown in Figure 4(d), is equal or lower than the negative of the voxel bandwidth (-16 Hz), then, several pixels from the original image are compressed into one pixel in the EPI. This situation occurs in problems with strong inhomogeneities, as shown in the example of Figure 4.

This singularity can be seen in the discrete system analyzing the singular values of the matrix $H_n$. We will observe the evolution of the singular values of $H_n$ by increasing the strength of the field map from a very weak inhomogeneity level to a stronger level. In figures 5 and 6 one can see the equivalent distortions caused in the EPI image. In this experiment we assume the GRE image is the correct one, and applied the distortion matrix constructed with the field map in 4(b). The strength of the original field map was reduced in order to observe the evolution of the distortions as the strength of the inhomogeneity increases.

We compared the singular values of the matrix produced with the standard discretization scheme against those of the matrix produced with our discretization scheme. We chose $R_y = 1$, in order to keep the matrices produced by different discretization schemes with the same dimensions (if $R_y \geq 2$ the proposed scheme leads to a system with non-null side blocks, as shown in equation (24)). For the proposed scheme, we chose $R_x = 3$, while the current scheme is the same as the proposed when $R_x = 1$.

In figures 7 and 8 one can observe the singular values of $H_n$ (current scheme) and $H_{n,n}$ (proposed scheme), with $n = M_y/2$, respectively. Observe that, as the inhomogeneity increases, the singular
values leave the unitary line (where all singular values are equal to one). The singular values with higher indexes go very fast to near zero values. Even though they are close to zero in the linear scale, in logarithm scale they are decreasing almost linearly. There is no significant difference in the distribution of the singular values for both models, as shown in figures 7 and 8. This means that the proposed model does not modify the conditioning of the problem. So we should expect similar behavior of the models in regions close to singularity, as the compressed regions.

This experiment illustrates the 3 points: a) when the inhomogeneity is not strong, as in figures 5(a) and 6(a), most of singular values are near the unitary line, and the conjugate phase method (transpose) provides acceptable results; b) as the inhomogeneity increases, intensity problems start to become significant, as in fig. 5(b) and 6(b), this can be seen as the singular values moving away from the unitary line, meaning that conjugate phase may no longer produce acceptable results, and a least squares approach becomes necessary; c) as the inhomogeneity gets still stronger, the system becomes very close to singularity, and even least squares solutions may fail to be acceptable due to ill-posedness. Therefore, regularization is required.

In figures 11 and 12 one can observe the results with the images from Figure 4. With \( R = 2 \) we start to see the attenuation effect of intravoxel inhomogeneity, which is not seen with \( R = 1 \). Also, the discretization artifacts are significantly reduced. We can also observe that the result with \( R = 3 \) is quite similar to \( R = 8 \), suggesting that values larger than \( R = 3 \) are not reasonable because of the computational cost.

3) Evaluating the Number of Side Blocks in the Approximated Distortion Matrix: Now we will evaluate the effect of the number of side blocks in the approximate matrix presented in (26). In section II-C, it was shown that the blocks of the distortion matrix are more significant near the diagonal, and the importance of the distant blocks decreases as \( |n_y - m_y| \) increases. Note that, the use of the side blocks is only required if \( R_y \geq 2 \).

In figures 15 and 16 we show the visual effect of using more side blocks. Observe that using just one block right and one block left of the main block diagonal already provides a significant improvement. The visual differences between the images with 4 and 16 side blocks are small, but noticeable. Note, that the image in Figure 16(d), produced with 16 side blocks, is very similar to the one in Figure

repeat the test with the full inhomogeneity strength of the field, and with half of the inhomogeneity strength. From figures 9 and 10 one can see that the error decreases very fast for low values of \( R \) with little increase in the computational cost, while for high \( R \) values the computational cost increases very fast and the error decreases very slowly.

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Fig. 13. Error and computational cost when increasing the number of side blocks, for half field strength.

Fig. 14. Error and computational cost when increasing the number of side blocks, for full field strength.

4(f), which uses full side blocks.

For this experiment we evaluate the performance of the matrix as the blocks far from the diagonal are removed. Again, we show, in Figure 13, the behavior of the error and the computational cost when increasing the number of blocks with half of the field strength, and in Figure 14, with full field strength. For this test we set $R = 3$. Observe that not too many blocks are necessary in order to reduce the error significantly. With 4 side blocks ($S = 2$) the normalized error is less than 0.5. The increase of the computational cost also prevents the use of too many blocks.

B. Reconstruction Simulations with the Proposed Discretization Model

Here, we compare the performance of the methods presented in Section III using the described discretization models with simulated EPI images. The methods are:

- Conjugate Phase (CP), as shown in equation (28).
- Least Squares (LS-CG), as shown in equation (31), minimized using conjugate gradient.
- Regularized Least Squares (RLS-TV), as shown in (33), with total variation as a regularization term, minimized using non-linear conjugate gradient [17].

The LS-CG and RLS-TV are iterative methods, the iterations were stopped when the improvement in quality was below $10^{-2}$ dB. The simulated EPI images were generated using (14), with the original images and known field maps, presented as $\gamma \Delta b(x)/2\pi$ in Figure 17. The acquisition time was 90 ms, the echo time 45 ms, and WCG noise was added to the simulated captured data, ensuring 50 dB of signal quality [19]. The quality of the reconstruction evaluated by the SNR [19], in dB, and the computational cost, including the cost of computing the matrices (M-CT) and the cost of the methods (CT), were considered. The results are shown in tables I and II and they are the average for several test problems. The visual results show three of the test problems. The discretization models will be compared under strong and medium inhomogeneity problems.

In tables I and II one can compare the difference in computational cost for each method, in (CT) the columns, and the time required to compute the matrices, in the (M-CT) column, which increases with $R$ and $S$. For $R_y = 1$, where the matrices remain with the same size for the current and the proposed model ($S$ is always equal to 0), we observe substantial gains using the proposed model with very little increase of computational cost, for all methods. This is a very motivating result. Also, the quality has a “jump” from $R_x = 1$ to $R_x = 2$, but it increases very little for $R_x \geq 2$ with respect to increase of cost.

When $R_y \geq 2$, the matrix requires an approximation in order to become computationally feasible. From the results in tables I and II we note that for the RLS-TV with: a) $S = 0$, the quality is not as good as using $R_y = 1$; b) $S = 1$, the quality is nearly the same as using $R_y = 1$, however there is some significant increase in the computational cost; and c) $S = 2$ (or even higher), the quality is much better.

The experiments were performed using MATLAB, using a x86 2GHz processor and 2GB (DDR2-666) memory.

Fig. 15. Visual effect of the proposed discretization scheme as the number of side blocks increases, with half field strength.

Fig. 16. Visual effect of the proposed discretization scheme as the number of side blocks increases, with full field strength.
better than using \( R_y = 1 \), however the computational cost is also high which is only justified if high quality is demanded. Gains obtained in the LS-CG and CP methods for \( R_y \geq 2 \) are small compared with the increase in computational cost, motivating the use of RLS-TV instead of LS-CG or CP if high quality is desired.

### TABLE I

**AVERAGE OF THE SNR, in dB, AND THE COMPUTATIONAL TIME\(^2\) OF THE MATRICES (M-CT) AND THE METHODS (CT), IN SECONDS, FOR HALF FIELD STRENGTH.**

<table>
<thead>
<tr>
<th>Scheme</th>
<th>RLS-TV</th>
<th>LS-CG</th>
<th>CP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current ( R = 1 ), ( S = 0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R_y = 2 ), ( R_x = 3 ), ( S = 0 )</td>
<td>17.5</td>
<td>13.0</td>
<td>15.1</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>20.9</td>
<td>15.2</td>
<td>17.9</td>
</tr>
<tr>
<td>( R_y = 2 )</td>
<td>21.0</td>
<td>15.7</td>
<td>17.9</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>21.0</td>
<td>15.3</td>
<td>17.9</td>
</tr>
<tr>
<td>( R_y = 2 )</td>
<td>20.3</td>
<td>15.3</td>
<td>17.9</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>21.3</td>
<td>15.3</td>
<td>17.9</td>
</tr>
<tr>
<td>( R_y = 2 )</td>
<td>20.3</td>
<td>15.3</td>
<td>17.9</td>
</tr>
</tbody>
</table>

### TABLE II

**AVERAGE OF THE SNR, in dB, AND THE COMPUTATIONAL TIME\(^2\) OF THE MATRICES (M-CT) AND THE METHODS (CT), IN SECONDS, FOR FULL FIELD STRENGTH.**

<table>
<thead>
<tr>
<th>Scheme</th>
<th>RLS-TV</th>
<th>LS-CG</th>
<th>CP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current ( R = 1 ), ( S = 0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R_y = 3 ), ( R_x = 1 ), ( S = 0 )</td>
<td>11.7</td>
<td>16.0</td>
<td>8.1</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>12.9</td>
<td>19.7</td>
<td>9.7</td>
</tr>
<tr>
<td>( R_y = 3 )</td>
<td>12.9</td>
<td>19.7</td>
<td>9.7</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>15.0</td>
<td>19.4</td>
<td>9.1</td>
</tr>
<tr>
<td>( R_y = 3 )</td>
<td>15.0</td>
<td>19.4</td>
<td>9.1</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>13.4</td>
<td>23.1</td>
<td>8.9</td>
</tr>
<tr>
<td>( R_y = 3 )</td>
<td>13.4</td>
<td>23.1</td>
<td>8.9</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>12.9</td>
<td>29.7</td>
<td>8.8</td>
</tr>
<tr>
<td>( R_y = 3 )</td>
<td>12.9</td>
<td>29.7</td>
<td>8.8</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>15.7</td>
<td>36.7</td>
<td>8.9</td>
</tr>
<tr>
<td>( R_y = 3 )</td>
<td>15.7</td>
<td>36.7</td>
<td>8.9</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>13.4</td>
<td>25.6</td>
<td>8.9</td>
</tr>
<tr>
<td>( R_y = 3 )</td>
<td>13.4</td>
<td>25.6</td>
<td>8.9</td>
</tr>
<tr>
<td>Proposed ( S = 0 )</td>
<td>13.4</td>
<td>24.3</td>
<td>9.0</td>
</tr>
<tr>
<td>( R_y = 3 )</td>
<td>13.4</td>
<td>24.3</td>
<td>9.0</td>
</tr>
</tbody>
</table>

Figures 17, 18, 19, and 20 show some visual results. In Figure 17 the original images, the field maps, and the simulated EPI images are shown in first to third columns respectively. The corrections with the standard model, \( R_x = 1 \), \( R_y = 1 \), and \( S = 0 \), are in Figure 18. In Figure 19 we observe the improvement in quality with the proposed model with \( R_x = 3 \), \( R_y = 1 \), and \( S = 0 \), while in Figure 20 the quality achieved by the proposed model with \( R_x = 3 \), \( R_y = 2 \), and \( S = 2 \).

One can notice from Figure 18(a)-(c) that conjugate phase does not correct intensity variations, that are better recovered with the least squared method, as shown in figures 18(d)-(f). One can also realize the advantage of using a regularized method, specially with total variation regularization. The regions that suffered compression, which are very close to singularity, were better recovered using RLS-TV, as in figures 18(g)-(i), but not with the other two methods.

For example, some internal structures (small lighted points) in the original Image 2 presented in Figure 17(d) were highly attenuated and compressed in the lower part of the object in the EPI image of Figure 17(f). However, observe that those internal structures were partially reconstructed in the RLS-TV images, presented in figures 18(h), 19(h), and 20(h). It is important to mention that those regions were recovered because the signal coming from them was not completely lost, only attenuated. If the signal is completely lost, it will not be recovered even by RLS-TV [17]. The other two methods fail to achieve this recovery. The CP method will not make it because it was not built to recover attenuated signals. The LS-CG method fails due to the lack of regularization. If one increases the number of iterations of the LS-CG, it starts to increase the noise level of the result before recovering the internal structures [18], making the results unusable.

Regarding the proposed model, the first improvement noticed within Figure 19 is the expanding regions, specially in the upper part of the object in the series from Image 1. These improvements come from the reduction of the discretization artifacts. Note, however, that the proposed model with \( R_y = 1 \), as well as the standard model, cannot provide a good recovery in regions of high horizontal derivative of the field map. The main effect of the high horizontal derivative in the field map is an intravoxel inhomogeneity that causes signal attenuation. This explains why some black spots in the object in figures 19(g)-(i) were not recovered. We expect to solve this problem using the proposed model with \( R_y \geq 2 \), shown in Figure 20.

In Figure 20, the improvements are, basically related with the recovery of the signal attenuation caused by the intravoxel inhomogeneity that occurs mainly in the y (horizontal) direction. Comparing the results from Figure 19 and from Figure 20, we notice that these distortions are more visible in the regions of higher horizontal derivative of the field map. Particularly, we can compare figures 20(g) with 19(g), 20(h) with 19(h), and 20(i) with 19(i), in order to observe that a substantial part of the black spots of signal attenuation was better recovered using \( R_y = 3 \) and \( S = 2 \).

### C. Reconstruction Experiments With Real Data

In this section we show some experiments comparing the proposed and the standard model using real data. We captured the data of a phantom that generates a strong inhomogeneous magnetic field. The MR scanner was an Elscint Prestige 2T. Figure 21 shows the GRE images, the estimated field maps and the EPI images. The acquisition parameters for the GRE images were: echo time of 10.999 milliseconds (ms) , resolution of 256\( \times \)144, FOV of 37.9 \( \times \)21.7 cm, the frequency encoding in horizontal direction, with 65.104 Hz of pixel bandwidth. The field maps were estimated from the GRE images using the method described in [11], [12], [13]. The echo times for the field map estimation were 10.999 ms and 11.999 ms. The resolution of the field maps is the same of the GRE images. The acquisition parameters for the EPI images were: echo time of 45.086 ms, resolution of 128\( \times \)72, the same FOV as the GRE, the slice thickness is 3 millimeters, the same as the GRE, and frequency encoding in the horizontal direction, with a 12.207 Hz of pixel bandwidth. For these experiments, the estimated field map has 2 times more resolution than for the EPI image, thus, for \( R \neq 2 \), it is necessary interpolation.

One can clearly notice the strong distortions caused by the inhomogeneous field map. The main problems are the geometric distortions and the intravoxel attenuation. Some visual results comparing the correction of the distorted EPI images are shown in Figure 22.

In Figure 22, one can notice the improvement with the use of the proposed model with real data. In figures 22(a)-(c), and 22(j)-(k) the results with the current model are shown and in figures 22(d)-(i), and 22(l) the results with the proposed one. The use of the proposed model with \( R_y = 3 \), \( R_x = 1 \), as shown in figures 22(d)-(f) and 22(l), provides some overall improvement over the standard model, specially in the regions where some discretization artifacts act, as in the upper part of the phantom. In figures 22(g)-(i), with the proposed model using \( R_y = 3 \), \( R_x = 3 \), and \( S = 2 \), one can notice the improvement in regions of high intravoxel inhomogeneity, mainly in the borders of the phantom.

Comparing the results in figures 22(b) and 22(c), and also 22(j) and 22(k), one can notice the difference between the unregularized least squares, minimized with conjugated gradient, and the regularized least squares using total variation regularization, both with the current
model. The same comparison can be made in figures 22(e) and 22(f), using the proposed model with $R_x = 3$, $R_y = 1$, and $S = 0$, as well as in figures 22(h) and 22(i), using the proposed model with $R_x = 3$, $R_y = 3$, and $S = 2$. One can notice that noise was significantly reduced using the total variation, compared to the unregularized LS-CG method. It is interesting to note that RLS-TV was not able to recover the signal coming from the lower part of the Image 2, shown in figures 22(k)-(l), as it partially did in the lower part of the Image 1, shown in figures 22(c), 22(f), and 22(i).

In figures 22(d)-(f) and 22(l), where the proposed model was utilized with $R_x = 3$, $R_y = 1$, and $S = 0$, one can notice the improvements in reducing discretization artifacts comparing with figures 22(a)-(c) and 22(j)-(k). In figures 22(h) and 22(i), where the proposed model was utilized with $R_x = 3$, $R_y = 3$, and $S = 2$, the observed improvements are the recovering of the signal attenuation caused intravoxel inhomogeneity and the reduction of the discretization artifacts.

V. CONCLUSIONS AND DISCUSSIONS

We proposed in this paper a new distortion model for solving inhomogeneity problems in EPI. This model leads to a new discretization scheme that provides better representation of the distortion process, reducing the discretization artifacts and including some intravoxel inhomogeneity. The increase in computational cost is very low if small values for $R$ and $S$ are chosen, for example $2 \leq R \leq 3$. 
and 0 \leq S \leq 2. Moreover, the increase in quality compensates the increase in the computational cost with the new approach.

The proposed model can be used with several distortion correction methods, such as conjugate phase, least squares, and regularized least squares. We observed improvement for all these methods. In particular, the regularized least squares was the one which obtained the best improvements in quality. The reason for these improvements is because regularized least squares provides a stable reconstruction for very ill-posed problems. It was observed a lowering of artifacts and some intravoxel inhomogeneity correction, besides the expected geometric distortion correction. Also, the contraction regions, which are more difficult to be restored, were well reconstructed with the help of the total variation penalty.

When \( R_y = 1 \), the proposed model leads to systems with the same size and nearly the same cost than the current discretization scheme, but with superior quality. This makes the new model, with \( R_y = 1 \), a good substitute of the current one for using for methods based on the solution of several small 1D systems, since these approaches are usually faster and less memory consuming.

The proposed model was developed for rectangular EPI problems; however, it might be extended for other fast imaging sequences, as spiral imaging. Also, we have performed experiments with EPI data provided by a 2T human MRI scanner. Even though we expect the model to perform similarly in different scanners, i.e. a 7T human MRI scanner, it is still an open problem.

In this paper we have not considered the effects of the slice direc-
VI. ACKNOWLEDGMENTS

The authors would like to thank the team of the Laboratory of Neuroimage at UNICAMP, specially to R. Covolan, G. Castellano, F. Pereira, A. Alessio, and M. Sercheli for their help in the data acquisition experiments and cooperation in the beginning of the MRI studies. Also, the authors are grateful for the technical cooperation and deep discussions provided by the team of the Laboratory of Image Reconstruction, at UNICAMP, specially to N. Cohen, E. Helou, R. Santos and E. Miqueles. The authors are also thankful to the anonymous referees for their constructive comments and suggestions.

APPENDIX

In this section we present the detailed development of the equations (9), (17), (18), (20), (21), and (22), showing the step-by-step mathematics to reach the proposed discrete model.
A. Extended Equations for the Current EPI Model

The step-by-step development for equation (9) is:

\[
\hat{s}[k_x, n_y] = \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] B[k_x, m_x; k_y, m_y] + \eta[k_x, k_y]
\]

\[
= \sum_{k_y = -M_y/2}^{M_y/2-1} \left\{ \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] B[k_x, m_x; k_y, m_y] \right\} + \eta[k_x, k_y]
\]

\[
= \sum_{k_y = -M_y/2}^{M_y/2-1} \left\{ \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] B[k_x, m_x; k_y, m_y] \right\} + \eta[k_x, k_y]
\]

\[
= \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] \left\{ \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{k_y = -M_y/2}^{M_y/2-1} \eta[k_x, k_y] e^{i2\pi \frac{k_y m_y}{M_y}} \right\}
\]

\[
= \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] \left\{ \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{k_y = -M_y/2}^{M_y/2-1} \eta[k_x, k_y] e^{i2\pi \frac{k_y m_y}{M_y}} \right\}
\]

\[
= \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] A[k_x, m_x; k_y, m_y] + \eta[k_x, k_y]
\]

(35)

where \( \delta[m_y - n_y] \) is the discrete impulse.

B. Extended Equations for the Proposed EPI Model

The step-by-step development for the proposed model in equation (17) is:

\[
\hat{s}[k_x, n_y] = \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] B[k_x, m_x; k_y, m_y]
\]

\[
= \sum_{k_y = -M_y/2}^{M_y/2-1} \left\{ \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] A[k_x, m_x; k_y, m_y] \right\} + \eta[k_x, k_y]
\]

\[
= \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] \left\{ \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{k_y = -M_y/2}^{M_y/2-1} \eta[k_x, k_y] e^{i2\pi \frac{k_y m_y}{M_y}} \right\}
\]

\[
= \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} f[m_x, m_y] A[k_x, m_x; n_y, n_y] + \eta[k_x, n_y]
\]

(36)

We can expand \( \hat{A}[k_x, m_x; n_y, n_y] \) in (36) and observe that:

\[
\hat{A}[k_x, m_x; n_y, n_y] = \sum_{k_y = -M_y/2}^{M_y/2-1} A[k_x, m_x; k_y, n_y] e^{i2\pi \frac{k_y m_y}{M_y}}
\]

\[
= \sum_{k_y = -M_y/2}^{M_y/2-1} \left\{ \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} \phi_x[x - R M_y, \phi_y[y - R M_y] \right\}
\]

\[
= \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} \left\{ \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{k_y = -M_y/2}^{M_y/2-1} \phi_x[x - R M_y, \phi_y[y - R M_y] \right\}
\]

\[
= \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} \left\{ \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{k_y = -M_y/2}^{M_y/2-1} \phi_x[x - R M_y, \phi_y[y - R M_y] \right\}
\]

\[
= \sum_{m_y = 0}^{M_y-1} \sum_{m_y = 0}^{M_y-1} \left\{ \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{k_y = -M_y/2}^{M_y/2-1} \phi_x[x - R M_y, \phi_y[y - R M_y] \right\}
\]

(37)

Which details the equations (17) and (18). Note that \( \varphi_y[y - R M_y] = \sum_{m_y = 0}^{M_y-1} \exp(-i2\pi k_y (y - R M_y)/(R M_y)) \), which results in a discrete version of the periodic sinc function, shown in (13).

The step-by-step development for the proposed model in equation (20) is:

\[
f'[n_x, n_y] = \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{m_y = 0}^{M_y-1} f[n_x, n_y] e^{i2\pi \frac{k_y n_y}{M_y}}
\]

\[
= \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{m_y = 0}^{M_y-1} f[n_x, n_y] \hat{A}[k_x, m_x; n_y, n_y] + \eta[k_x, n_y]
\]

\[
= \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{m_y = 0}^{M_y-1} f[n_x, n_y] \left\{ \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{k_y = -M_y/2}^{M_y/2-1} \eta[k_x, k_y] e^{i2\pi \frac{k_y n_y}{M_y}} \right\}
\]

\[
= \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{m_y = 0}^{M_y-1} f[n_x, n_y] \left\{ \sum_{k_y = -M_y/2}^{M_y/2-1} \sum_{k_y = -M_y/2}^{M_y/2-1} \eta[k_x, k_y] e^{i2\pi \frac{k_y n_y}{M_y}} \right\}
\]
\[ f'(n_x, n_y) = \sum_{m_y=0}^{M_y-1} \sum_{m_x=0}^{M_x-1} f(m_x, m_y) H(n_x - m_x, n_y - m_y) + f(n_x, n_y) \]

where

\[ H(n_x, m_x; n_y, m_y) = \sum_{k_x = -M_x}^{M_x-1} \sum_{k_y = -M_y}^{M_y-1} \tilde{A}[k_x, m_x; n_y, m_y] e^{2\pi i \frac{k_x n_x}{M_x}} e^{2\pi i \frac{k_y n_y}{M_y}} \]

\[ = \sum_{k_x = -M_x}^{M_x-1} \sum_{k_y = -M_y}^{M_y-1} \frac{1}{R^2} \sum_{x=0}^{R M_x-1} \sum_{y=0}^{R M_y-1} \phi_x[x - R m_x] \phi_y[y - R m_y] \]

\[ \times \varphi_y[y - R n_y] e^{-i \pi x^2 / R M_x} \]

\[ = \frac{1}{R^2} \sum_{y=0}^{R M_y-1} \phi_y[y - R n_y] \varphi_y[y - R n_y] \sum_{x=0}^{R M_x-1} \phi_x[x - R m_x] \]

\[ \times \sum_{k_x = -M_x}^{M_x-1} \sum_{k_y = -M_y}^{M_y-1} \frac{1}{R^2} \sum_{x=0}^{R M_x-1} \sum_{y=0}^{R M_y-1} \phi_x[x - R m_x] \phi_y[y - R m_y] \]

\[ = \frac{1}{R^2} \sum_{y=0}^{R M_y-1} \phi_y[y - R n_y] \varphi_y[y - R n_y] \tilde{H}(n_x, m_x; y) \]

Which details the equations (20), (21), and (22).

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