see in both cases the effect of over-modeling. Given the prior process knowledge, the model $y(t) = c$ with $\hat{c} = 5.15$ seems to be the most appropriate model.

9.3 Experimental Data

In addition to an evaluation of the parameter estimates with respect to prior knowledge and of the model simulations and predictions, in this section we will explicitly use the residuals for model validation. Recall that the residuals are defined as

$$\varepsilon(t) := y(t) - \hat{y}(t|\vartheta)$$  \hspace{1cm} (9.4)

The residuals do play a key role in the model validation process, since they reflect the difference between the measured output $y(t)$ and the predicted model output $\hat{y}(t|\vartheta)$, given an estimate of $\vartheta$. In the following, we therefore introduce some common residual tests and finalize this section with a real-world example.

9.3.1 Graphical Inspection

A first test on the residuals should be based on graphical inspection. Plotting the residuals, at an appropriate scale, will directly show some peculiarities, as outliers, drift and periodicity, see Fig. 5.4 and Fig. 7.5 in previous chapters. Let us also demonstrate this graphical inspection to the moving object of Example 7.3.

Example 9.6 Moving object (constant velocity): Recall that the following observations on an object moving in a straight line with constant velocity $v$, as presented in Table 9.3, were available.
The proposed model for the moving object was
\[
y(t) = s_0 + vt + \epsilon(t)
\]
with final estimates \(\hat{s}_0 = 5.7027\) ft and \(\hat{v} = 4.0215\) ft/s. In Fig. 9.6 the residuals \(\epsilon(t) := y(t) - \phi(t)^T \hat{\theta}\), with \(\theta = [s_0 \ v]^T\), are plotted.

Notice from Fig. 9.6 that the observation at time index 4 is possibly an outlier. Furthermore, the residuals show some periodicity, but the time series is far too short to come up with firm statements. Hence, this clearly illustrates the problem of model validation for small data sets.

However, apart from the model validation problem related to small data sets, it is never clear beforehand whether drift and periodicity in the residuals originate from an invalid model; the experimental data may contain these characteristics as well. Hence, analysis of the experimental data, using, for instance, linear regression and correlation techniques, and examination of the sensor system may help to solve this dilemma. Furthermore, some basic properties of the prediction error sequence, as \(\max_t |\epsilon(t)| = \|\epsilon\|_\infty\) and \(\frac{1}{N} \sum_{t=1}^{N} \epsilon(t)^2 = \|\epsilon\|_2^2\), may also help to validate the model. For instance, when \(\|\epsilon\|_\infty\) is large, most likely outliers are present in the data, and thus, for an appropriate validation of the model, these should be removed. The 2-norm of the residuals can be used to compare models, and, under the assumption that the system is time-invariant, it indicates the expected magnitude of prediction errors. Hence, on the basis of these statistics, interpreted as quantities calculated from a set of data, one may or may not accept the model as valid.
9.3.2 Correlation Tests

Ideally, the residuals or prediction errors related to dynamic models should not depend on the inputs or previous residuals. If that is not the case, there is room for model improvement. For instance, in case of a general transfer function model structure, the exogenous part \( G(q)u(t) \) can be extended with delayed inputs or the noise model \( H(q)e(t) \) modified. To check the dependencies, it is very natural to study the correlations between residuals and past inputs. Let \( N \) data points of the input and residuals, respectively, be given. Then, the cross-correlation function, see also Sect. 4.1, between input and residuals is given by

\[
r_{ue}(l) = \frac{1}{N-l} \sum_{i=1}^{N-l} u(i)\varepsilon(i+l)
\]

(9.5)

Hence, if the cross-correlations are small, this indicates that the residuals, and thus the model output \( y(t) \), do not contain any further information originated from past inputs. In particular, it should be noted that significant correlation for negative \( l \) indicates output feedback in the input.

In a similar way, we can use the auto-correlation function for investigating the correlations among the residuals. The autocorrelation function is given by

\[
r_{ee}(l) = \frac{1}{N-l} \sum_{i=1}^{N-l} \varepsilon(i)\varepsilon(i+l)
\]

(9.6)

As mentioned before, the auto-correlation function can be used to test whether the residuals are white and thus do not contain any further information that can be used to improve the model predictions. A popular test for whiteness of the residuals, implicitly assuming that the residuals are normally distributed and within a range of \( M \) data points, is

\[
\frac{N}{\hat{r}_{ee}(0)^2} \sum_{l=1}^{M} \hat{r}_{ee}(l)^2 \leq \chi^2_\alpha(M)
\]

(9.7)

with \( \chi^2_\alpha(M) \) the \( \alpha \)-level of the \( \chi^2(M) \)-distribution (see Appendix B). Hence, if this inequality holds, we may conclude that the residuals are serially uncorrelated over a range of \( M \) data points.

For a formal test on the statistically independence between residuals and inputs, we could check if the following holds for the estimated cross-correlations:

\[
|\hat{r}_{ue}(l)| \leq \sqrt{\frac{P_1}{N} N_\alpha}
\]

(9.8)

where \( P_1 = \sum_{i=-\infty}^{\infty} r_{ei}(l)r_{iu}(l) \), and \( N_\alpha \) denotes the \( \alpha \)-level of the standard normal distribution, \( N(0, 1) \). Notice that, since the right-hand side of (9.8) does not depend
on \( l \), it is a constant. Apart from this formal test, we could also investigate the scatter plot of the pairs \((\varepsilon(t), u(t - l))\).

Let us demonstrate the correlation tests on different models of a mass-spring-damper system with known input and output.

**Example 9.7 Mass-spring-damper:** Let in discrete-time, the model of a mass-spring-damper system be given by

\[
\begin{align*}
x(t) &= \begin{bmatrix} 1 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} x(t - 1) + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} u(t - 1) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\end{align*}
\]

The corresponding discrete-time transfer function is

\[
G(q) = \frac{0.25q^{-2}}{1 - 1.5q^{-1} + 0.75q^{-2}}
\]

If, however, we select an ARX(1, 1, 0) and ARX(2, 1, 0) model structure, we obtain the discrete-time transfer functions

\[
G_1(q) = \frac{0.1337}{1 - 0.9082q^{-1}}
\]

and

\[
G_2(q) = \frac{0.0762}{1 - 1.6979q^{-1} + 0.8735q^{-2}}
\]

respectively. The corresponding residuals and correlation functions are presented in Figs. 9.7 and 9.8.

From Fig. 9.7 significant correlations between the residuals can be seen, indicating an inappropriate model structure. Furthermore, the cross-correlation function between input and residuals shows a clear peak at lag 2, which refers to a deficiency in the time lag of the model. Increasing the model complexity toward an ARX(2, 1, 0) model removes the significant correlations between the residuals, but the peak at lag 2 in the cross-correlation function remains, as expected. Obviously, an ARX(2, 1, 2) model, with a similar structure as the transfer function \(G(q)\) derived from the discrete-time state-space model, gives a perfect fit.

However, unlike the previous example, in practice always some noise is present. For an evaluation of a noisy data case, the heating system (Example 2.2) is considered again.

**Example 9.8 Heating system:** The following responses to a random binary signal with switching probability \((p_0)\) of 0.2 and 0.5, respectively, have been measured from a simple heating system (see Figs. 9.9 and 9.10). Recall that the input of the
Fig. 9.7 Residuals, correlation functions, and $\alpha$-levels (see (9.7) and (9.8)) related to ARX(1, 1, 0) model

system is the voltage applied to the heating element and the output, also in voltage, is measured with a thermistor. The maximum allowable magnitude of the input is 10 V, and the sampling interval is 0.08 s.
In both figures, the effect of the initial conditions is clearly visible. Furthermore, apart from the difference in switching probability, a similar behavior is seen. In an identification step, after neglecting the first 2 s of the data set and after detrending both the input and output signals, we found from the first data set, with $p_0 = 0.2$,
9.3 Experimental Data

Fig. 9.11 Auto- and cross-correlation functions related to an ARX(2, 1, 3) model for $p_0 = 0.2$

Fig. 9.12 Auto- and cross-correlation functions related to an ARX(2, 1, 3) model for $p_0 = 0.5$
the following ARX model:

\[ G(q) = \frac{0.0507q^{-3}}{1 - 1.437q^{-1} + 0.520q^{-2}} \]
with a loss function value of 0.00181512 and an FPE function value of 0.00182725. In what follows, we fix this model structure and test it on the two data sets. The correlation functions related to both data sets are presented in Figs. 9.11 and 9.12. In both cases significant auto-correlations between the residuals at lag 1 can be seen. Furthermore, for lag 2–5, also significant cross-correlations between input and residuals are visible, indicating an additional time lag of 2, i.e., 0.16 seconds. Hence, at this point we conclude that there is some model deficiency.

At last, we evaluate the model behavior in time domain by simulating the ARX(2, 1, 3) model for both data sets. The results are presented in Figs. 9.13 and 9.14. Since the model shows a good behavior with respect to the observed data, our overall conclusion is that the model is appropriate, at least for short-term predictions. Thus, as yet, the ARX(2, 1, 3) model passes the model validation test.

Finally, in this subsection, we will demonstrate the use of predictions and experimental data in a cross-validation step by a real-world example.

Example 9.9 Storage facility (based on [KD09]): A discrete-time nonlinear model describing the temperature dynamics in a storage room with a respiring product and suitable for incorporation in a model-based control strategy is given by (see [KPL03])

\[
T_p(t) = \left( p_1 + \frac{p_2}{p_3 + p_4 u(t-1)} + \frac{p_5}{p_6 + p_7 u(t-1)} \right) T_p(t-1) + \frac{p_8 + p_9 u(t-1)}{p_3 + p_4 u(t-1)} T_e(t-1) + \frac{p_{10} + p_{11} u(t-1)}{p_6 + p_7 u(t-1)} X_e(t-1) + \left( p_{12} + \frac{p_{13}}{p_6 + p_7 u(t-1)} \right)
\]

(9.9)

where \( T_p(t) \) is measured. The variable \( T_p \) denotes the temperature of the produce (°C), \( T_e \) is the external temperature (°C), \( X_e \) is the external absolute humidity (kg/kg), and \( p = [p_1, \ldots, p_{13}]^T \) the parameter vector. Finally, the control input \( u \) denotes the product of fresh inlet ratio and ventilation rate and is bounded by \( 0 \leq u \leq 1 \). In Fig. 9.15 a schematic representation of the storage facility with corresponding variables is presented. The variables \( T_{in} \) (air temperature in channel), \( T_a \) (air temperature in bulk), \( X_{in} \) (absolute humidity in channel), and \( X_a \) (absolute humidity in bulk), as shown in the figure, do not appear in (9.9) as a result of a model reduction step based on singular perturbation analysis of the full system, see [KPL03] for details on this. In this model reduction step, quasi-steady states of air temperature and humidity were substituted in the heat balance of the product. This substitution finally leads to the rational terms, as in (9.9), and it enforces that the product temperature in (9.9) depends only on the external temperature \( T_e \) and external absolute humidity \( X_e \).